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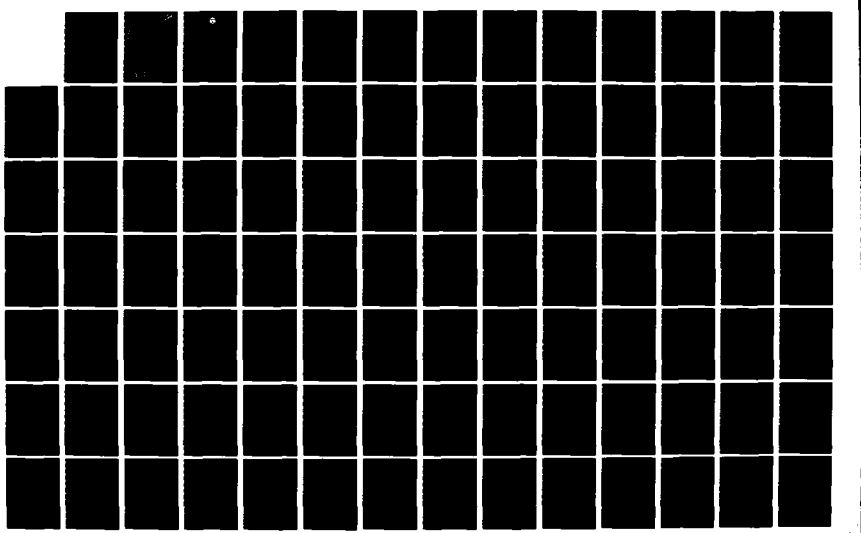

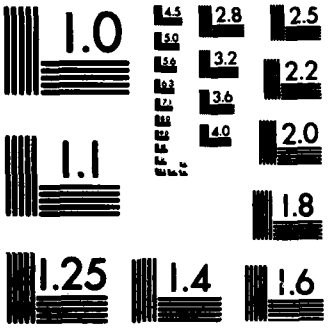

NOSC TD 610

Technical Document 610

XWVG: A WAVEGUIDE PROGRAM FOR TRILINEAR TROPOSPHERIC DUCTS

Computer program for calculating the signal levels of EM waves propagating over seawater

GB Baumgartner Jr

30 June 1983

Period of work: October 1982 – June 1983

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XWVG is a reliable waveguide program for calculating the coherent and incoherent mode sum values of the electromagnetic field strength in dB relative to the free space value, for signals propagating in a trilinear tropospheric duct over seawater. XWVG allows for the effects of surface roughness, absorption by atmospheric gases, and the variation of the complex index of refraction of seawater as a function of temperature, salinity, and frequency. XWVG has been successfully used in the frequency range 50 MHz to 94 GHz and should be usable at higher frequencies. XWVG uses the root-finding routine of Shellman and Morfitt to find the complex modes propagating in the duct. This guarantees that all modes with an attenuation rate below a prespecified level will be found. A description of XWVG is provided, along with a complete program listing and sample output.

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CONTENTS

1. INTRODUCTION

This document is a description of XWVG, a program for calculating the signal levels of electromagnetic waves propagating in a tropospheric ducting environment over seawater. This program is based on the waveguide formulation of the tropospheric ducting problem given by Marcus and Stuart [1, 2]. Like the program DUCT developed by Marcus, Stuart, and Ellis [2, 3], XWVG uses the root-finding program of Shellman and Morfitt [4] to find the complex modes that propagate in the tropospheric duct.

XWVG has been developed for the purpose of providing a program which guarantees that all waveguide modes with attenuation rates below a specified value will be found. XWVG is particularly useful for conducting case studies over a large dynamic range of frequencies, and its results can serve as a standard against which results of quicker but more approximate methods [5] may be measured.

XWVG assumes that the modified refractivity profile of the troposphere can be successfully modeled with three linear segments. For many naturally occurring tropospheric ducts this is a good assumption. The program DUCT of Marcus, Stuart, and Ellis allows the modified refractivity profile to be modeled with up to five linear segments and hence can describe a wider variety of refractivity profiles.

The waveguide modes that can propagate in a tropospheric ducting environment are the complex zeros of a transcendental function. In the Marcus and Stuart formulation of the tropospheric ducting problem this modal function is expressed as a determinant. In XWVG, where the modified refractivity profile is restricted to three linear segments, the modal function is evaluated by expanding the determinant and numerically evaluating the resulting expression. This allows the derivative of the modal function, which is needed by the Shellman and Morfitt root-finding program and for the evaluation of the modal excitation factors, to be evaluated easily. In DUCT, the modal function is evaluated by numerically evaluating the determinant. Because of the method used in DUCT to evaluate the derivative of the modal

function, DUCT is restricted to refractivity profiles with five or fewer linear segments.

XWVG includes an approximate allowance for surface roughness and atmospheric absorption, effects which are not included in DUCT. XWVG also includes a subroutine that evaluates the complex index of refraction of seawater as a function of frequency, temperature, and salinity.

The major advantage that DUCT and XWVG have over some other implementations of tropospheric ducting models is the use of the complex root-finding program of Shellman and Morfitt. Use of this routine guarantees that all modes with an attenuation rate below a prespecified level will be found. However, this advantage does not come without a price. The Shellman and Morfitt root-finding routine has an extremely long execution time compared with other less infallible methods of finding modes [6].

2. MATHEMATICAL FORMULATION OF THE MODEL

MODIFIED REFRACTIVITY PROFILE

The index of refraction of the atmosphere is given by

$$n(\bar{r}) = [\varepsilon_r(\bar{r})\mu_r(\bar{r})]^{\frac{1}{2}} , \tag{2.1}$$

where ε_r and μ_r are the relative permittivity and permeability of the atmosphere. Throughout this document the assumption is made that

$$\mu_r(\bar{r}) = 1 , \tag{2.2}$$

and the only variation of the index of refraction is with height above the surface of the earth.

When modeling tropospheric propagation, it is mathematically convenient to utilize a rectangular coordinate system to describe atmospheric wave propagation over a curved earth. The curvature of the earth may be approximated by modifying the index of refraction of the atmosphere [7, 8]. This modification is accomplished by adding a term to n(z), where z is the height above the surface of a flat earth, which would cause a ray to bend in such a way that its height above the flat earth at each point would be the same as if it were a straight ray over a curved earth. The modified index of refraction is then given by

$$m(z) = n(z) + \frac{z}{a} , \tag{2.3}$$

where a is the radius of the earth.

The accuracy of the flat-earth approximation depends upon the frequency of the electromagnetic wave and the height above the surface of the earth at which the signal level is measured [9, 10]. The approximation breaks down as the frequency and height above the earth's surface increase. For the standard atmosphere, Pekeris [9] finds that the fractional error in the height-gain functions is approximately

$$\Delta = \frac{2\pi\sqrt{6}}{5} \frac{h^{5/2}}{\lambda a^{3/2}} \tag{2.4}$$

$$= 1.911 \times 10^{-10} \frac{[h(m)]^{5/2}}{\lambda(m)} \ . \tag{2.5}$$

From this it can be seen that for signals with wavelength λ = 3 cm (f = 10 GHz) at a height of h = 1000 m the fractional error in the height-gain function is approximately 20%.

The modified index of refraction of the atmosphere, m(z), is very nearly one, the difference being on the order of 10^{-3}. For this reason it is convenient to introduce the modified refractivity, M(z), defined by

$$M(z) = [m(z) - 1] \times 10^{6} \ . \tag{2.6}$$

Often, the modified refractivity can be approximated by three linear segments, as shown in figure 1. The model developed in this document assumes that the modified refractivity can be approximated with three straight line segments and that this one profile is valid for all range values, x. An additional assumption made is that the slope, $\frac{dM}{dz}$, of the modified refractivity profile is positive in the first and third layers. In the second layer, $\frac{dM}{dz}$ can be either positive or negative, but not zero.

Let $M_i(z)$ denote the value of the modified refractivity in the i-th layer. The modified refractivity profile shown in figure 1 can then be written

$$M_1(z) = M(z_{ref}) + \frac{dM_1}{dz}(z - z_{ref}) \qquad (0 \leq z \leq z_2), \tag{2.7}$$

$$M_2(z) = M_1(z_2) + \frac{dM_2}{dz}(z - z_2) \qquad (z_2 \leq z \leq z_3), \tag{2.8}$$

and

$$M_3(z) = M_2(z_3) + \frac{dM_3}{dz}(z - z_3) \qquad (z_3 \leq z) \ . \qquad (2.9)$$

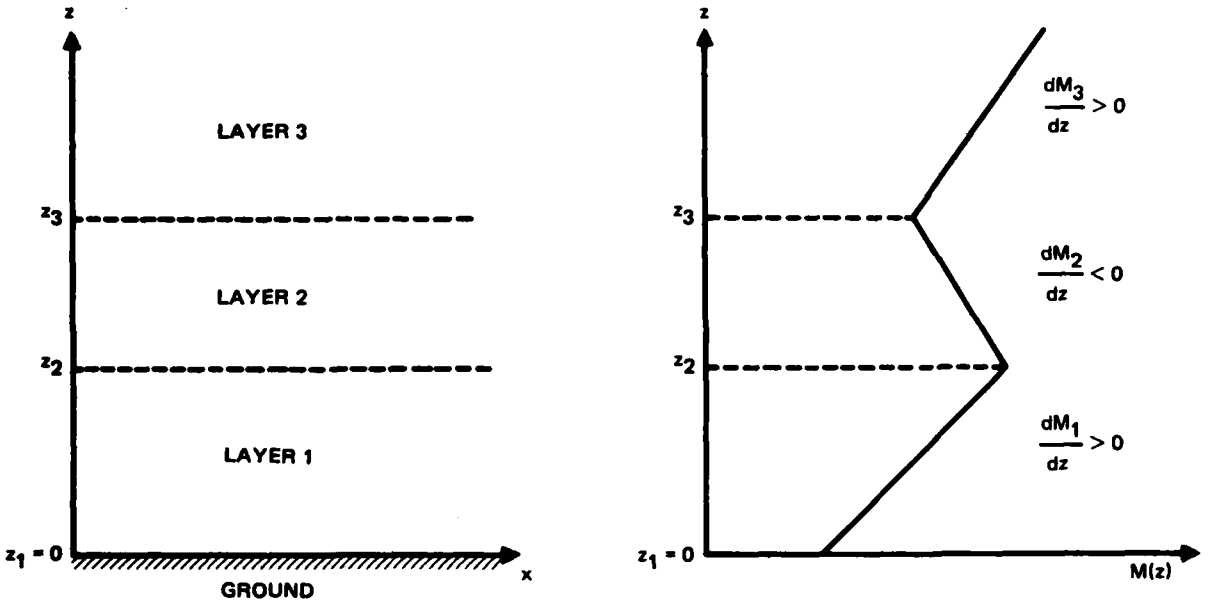

Figure 1. Trilinear modified refractivity profile. The slope of M(z) is assumed to be positive in the first and third layers. In the second layer the slope can be positive or negative.

The square of the modified index of refraction is then

$$m_1^2(z) \cong 1 + 2 \times 10^{-6} M_1(z) + i\eta$$
$$\cong 1 + \alpha_1(z - H) + i\eta \qquad (0 \leq z \leq z_2), \qquad (2.10)$$

$$m_2^2(z) \cong 1 + 2 \times 10^{-6} M_2(z) + i\eta$$
$$\cong 1 + \alpha_1(z_2 - H) + \alpha_2(z - z_2) + i\eta \qquad (z_2 \leq z \leq z_3), \qquad (2.11)$$

and

$$m_3^2(z) \cong 1 + 2 \times 10^{-6} M_3(z) + i\eta$$
$$\cong 1 + \alpha_1(z_2 - H) + \alpha_2(z_3 - z_2)$$
$$+ \alpha_3(z - z_3) + i\eta \qquad (z_3 \leq z) \ . \qquad (2.12)$$

The slopes α_i are related to the slopes $\frac{dM_i}{dz}$ by

$$\alpha_i = 2 \times 10^{-6} \frac{dM_i}{dz} . \tag{2.13}$$

The term $i\eta$ introduced in equations (2.10-2.12) allows for attenuation of the signal due to absorption by atmospheric gases. The height H is chosen so that at some reference height, z_{ref}, in the first layer, $M(z_{ref})$ has a specified value. H is related to α_1 and $M(z_{ref})$ by

$$H = z_{ref} - \frac{2 \times 10^{-6} M(z_{ref})}{\alpha_1} . \tag{2.14}$$

The complex index of refraction of the ground is assumed to be independent of position and given by

$$n_g^2 = \varepsilon_g - i \frac{\sigma_g}{\varepsilon_o \omega} , \tag{2.15}$$

where ε_g is the relative permittivity of the ground, σ_g is the ground conductivity, ε_o is the permittivity of free space, and $f = \frac{\omega}{2\pi}$ is the frequency.

HORIZONTALLY POLARIZED WAVE PROPAGATION

Horizontally polarized waves may be considered as due to a radiating magnetic dipole $\bar{p}$ oriented in the z-direction and located at $x = y = 0$, $z = z_T$. In a laterally homogeneous medium the fields due to such a dipole exhibit axial symmetry and may be obtained from the magnetic Hertz potential vector [11]

$$\bar{\Pi}(r,z) = \Pi(r,z)\bar{e}_z , \tag{2.16}$$

where $\bar{e}_z$ is the unit vector in the z-direction and r is the lateral distance from the source location. A time dependence of $e^{+i\omega t}$ is assumed. The

electric and magnetic fields are related to the magnetic Hertz potential vector by

$$\begin{aligned}\bar{D} &= -i\mu\varepsilon\omega\bar{\nabla} \times \bar{\Pi} \\ &= -i\varepsilon\mu_o\omega\bar{\nabla} \times \bar{\Pi}\end{aligned} \tag{2.17}$$

and

$$\bar{H} = \bar{\nabla} \times \bar{\nabla} \times \bar{\Pi} \ . \tag{2.18}$$

The Hertz potential may be expressed as [11]

$$\Pi(r,z) = \frac{1}{4\pi} \int_C \rho d\rho \ H_o^{(2)}(r\rho)\tilde{\Pi}(\rho,z) \ , \tag{2.19}$$

where $H_o^{(2)}(\xi)$ is the zeroth-order Hankel function of the second kind and the contour C is shown in figure 2. In the j-th atmospheric layer (j = 1,2,3) the function $\tilde{\Pi}(\rho,z)$ satisfies the differential equation

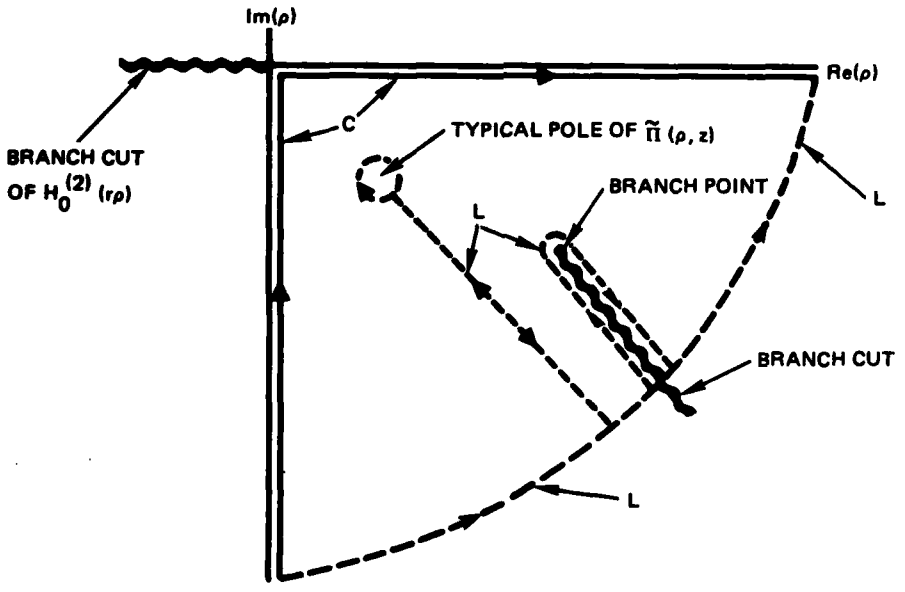

Figure 2. Contour of integration C for evaluation of equation (2.19). By Cauchy's Theorem the contour L is equivalent to the contour C. The branch cut in the fourth quadrant is due to the factor $\sqrt{n_g^2 - \rho^2/k^2}$ introduced into $\tilde{\Pi}$ because of the ground boundary condition.

$$\left\{\frac{d^2}{dz^2} + k^2\left[m_j^2(z) - \frac{\rho^2}{k^2}\right]\right\} \tilde{\Pi}_j(\rho,z) = -p\delta(z - z_T) \ , \tag{2.20}$$

where k is the free space wave number and $\delta(\xi)$ is the Dirac delta function. In the ground, $\tilde{\Pi}(\rho,z)$ satisfies the differential equation

$$\left\{\frac{d^2}{dz^2} + k^2\left[n_g^2 - \frac{\rho^2}{k^2}\right]\right\} \tilde{\Pi}_g(\rho,z) = 0 \ . \tag{2.21}$$

Continuity of the tangential components of the $\bar{E}$ and $\bar{H}$ fields yields the continuity conditions [12]

$$\tilde{\Pi}_g(\rho,o) = \tilde{\Pi}_1(\rho,o) \ , \tag{2.22}$$

$$\frac{d}{dz}\{\tilde{\Pi}_g(\rho,o)\} = \frac{d}{dz}\{\tilde{\Pi}_1(\rho,o)\} \ , \tag{2.23}$$

$$\tilde{\Pi}_j(\rho,z_{j+1}) = \tilde{\Pi}_{j+1}(\rho,z_{j+1}) \qquad (j = 1,2) \ , \tag{2.24}$$

and

$$\frac{d}{dz}\{\tilde{\Pi}_j(\rho,z_{j+1})\} = \frac{d}{dz}\{\tilde{\Pi}_{j+1}(\rho,z_{j+1})\} \qquad (j = 1,2) \ . \tag{2.25}$$

The boundary conditions that $\tilde{\Pi}(\rho,z)$ must satisfy are: as $z \to -\infty$, $\tilde{\Pi}_g(\rho,z)$ must represent an outgoing exponentially decaying wave; and as $z \to +\infty$, $\tilde{\Pi}_3(\rho,z)$ must represent an outgoing wave.

The solution to equation (2.21) is

$$\tilde{\Pi}_g(\rho,z) = A_g(\rho)e^{i\gamma z} \ , \tag{2.26}$$

where

$$\gamma = k[n_g^2 - \beta^2]^{\frac{1}{2}} \tag{2.27}$$

and

$$\beta = \frac{\rho}{k} \quad . \tag{2.28}$$

With time dependence $e^{+i\omega t}$, equation (2.26) represents an outgoing wave as $z \to -\infty$. If the branch of the square root in equation (2.27) is chosen so that

$$\mathrm{Im}(\gamma) < 0 \ , \tag{2.29}$$

then as $z \to -\infty$, $|\tilde{\Pi}_g(\rho,z)|$ decays exponentially.

The complete solution of equation (2.20) will be the sum of the general solution of the homogeneous equation corresponding to (2.20) and a particular solution of the inhomogeneous equation. With the new independent variables

$$q_j(z) = \left(\frac{k}{|\alpha_j|}\right)^{2/3} [m_j^2(z) - \beta^2] \qquad (j = 1,2,3) \ , \tag{2.30}$$

the homogeneous equation corresponding to (2.20) can be written as

$$\left\{\frac{d^2}{dq_j^2} + q_j\right\} \tilde{\Pi}_j(\rho,q_j) = 0 \qquad (j = 1,2,3) \ . \tag{2.31}$$

This is Stokes' differential equation and its solution can be written as

$$\tilde{\Pi}_j(\rho,q_j) = A_j(\rho) f_j(q_j) + B_j(\rho) g_j(q_j) \qquad (j = 1,2,3) \ , \tag{2.32}$$

where $f_j(\xi)$ and $g_j(\xi)$ are any two linearly independent combinations of Airy functions. The functions $f_j(\xi)$ and $g_j(\xi)$ need not be the same combinations of Airy functions in each atmospheric layer. For numerical purposes it is convenient to choose [2]

$$f_j(\xi) = k_1(\xi) \qquad (j = 1,2) \ , \tag{2.33}$$

$$f_3(\xi) = h_1(\xi) - e^{-4\pi i/3} h_2(\xi) \ , \tag{2.34}$$

$$g_j(\xi) = k_2(\xi) \qquad (j = 1,2) \ , \tag{2.35}$$

and

$$g_3(\xi) = h_2(\xi) \ , \tag{2.36}$$

where the modified Hankel functions [13], $h_1(\xi)$ and $h_2(\xi)$, and the functions $k_1(z)$ and $k_2(z)$ are related to each other and the Airy functions through

$$\begin{aligned} k_1(z) &= h_1(z) \\ &= -2i(12)^{1/6}\mathrm{Ai}(-ze^{2\pi i/3}) \ , \end{aligned} \tag{2.37}$$

$$\begin{aligned} k_2(z) &= h_2(z) - e^{4\pi i/3}h_1(z) \\ &= 2(12)^{1/6}e^{\pi i/6}\mathrm{Ai}(-z) \ , \end{aligned} \tag{2.38}$$

and

$$h_2(z) = 2i(12)^{1/6}\mathrm{Ai}(-ze^{-2\pi i/3}) \ . \tag{2.39}$$

Equation (2.32) represents the solution $\tilde{\Pi}(\rho,z)$ for every atmospheric layer except the one the transmitter is in. That is, only for the layer containing $z = z_T$ is equation (2.20) inhomogeneous. Assuming the transmitter to be located in the M-th layer (M = 1,2,3), two particular solutions to (2.20) are [1, 2]

$$\tilde{\Pi}_M^{(P)}(\rho,q_M) = \begin{cases} R_M k_1(q_{M<})k_2(q_{M>}) & (M = 1,2) \\ R_M[h_1(q_{M<}) - e^{-4\pi i/3}h_2(q_{M<})]h_2(q_{M>}) & (M = 3) \end{cases} \tag{2.40}$$

and

$$\tilde{\Pi}_M^{(P)}(\rho,q_M) = \begin{cases} -R_M k_1(q_{M>})k_2(q_{M<}) & (M = 1,2) \\ -R_M[h_1(q_{M>}) - e^{-4\pi i/3}h_2(q_{M>})]h_2(q_{M<}) & (M = 3) \ . \end{cases} \tag{2.41}$$

In these equations

$$q_{M<} = q_M(\min[z,z_T]) \ , \tag{2.42}$$

$$q_{M>} = q_M(\max[z,z_T]) \ , \tag{2.43}$$

and

$$R_M = \frac{p}{Wq'_M} \ . \tag{2.44}$$

In equation (2.44)

$$q'_M = \frac{\partial q_M}{\partial z} = \left(\frac{k}{|\alpha_M|}\right)^{2/3} \alpha_M \tag{2.45}$$

and W is the Wronskian

$$W = f_j(\xi)g'_j(\xi) - f'_j(\xi)g_j(\xi) \ . \tag{2.46}$$

For the choice of solutions (2.33-2.36), the Wronskian has the constant value

$$W = -\frac{4i}{\pi}\left(\frac{3}{2}\right)^{1/3} \tag{2.47}$$

in all atmospheric layers.

Combining the general solution (2.32) of the homogeneous equation (2.31) with either one of the particular solutions (2.40, 2.41) of the inhomogeneous equation (2.20) yields

$$\tilde{\Pi}_j(\rho,z) = A_j(\rho)k_1(q_j) + B_j(\rho)k_2(q_j) + \tilde{\Pi}_M^{(P)}(\rho,q_M)\delta_{jM} \quad (j = 1,2) \tag{2.48}$$

and

$$\tilde{\Pi}_3(\rho,z) = A_3(\rho)[h_1(q_3) - e^{-4\pi i/3}h_2(q_3)] + B_3(\rho)h_2(q_3)$$

$$+ \tilde{\Pi}_M^{(P)}(\rho,q_M)\delta_{3M} \ , \tag{2.49}$$

where δ_{jM} is the Kroneker delta

$$\delta_{jM} = \begin{cases} 1 & (j = M) \\ 0 & (j \neq M) \ . \end{cases} \tag{2.50}$$

For the assumed time dependence, $e^{i\omega t}$, the function (2.49) will represent an outgoing wave as $z \to +\infty$ if

$$A_3(\rho) = 0 \ . \tag{2.51}$$

The other coefficients A_j and B_j are determined from the continuity conditions (2.22-2.25). Thus, the coefficients A_j and B_j are solutions of the system of equations

$$A_1\hat{k}_1(q_{11}) + B_1\hat{k}_2(q_{11}) = \frac{i\gamma}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\tilde{\Pi}_1^{(P)}(\rho,q_{11})\delta_{1M}$$

$$- \frac{1}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\frac{d}{dz}\left\{\tilde{\Pi}_1^{(P)}(\rho,q_{11})\right\}\delta_{1M} \ , \tag{2.52}$$

$$A_1k_1(q_{12}) + B_1k_2(q_{12}) - A_2k_1(q_{22}) - B_2k_2(q_{22})$$

$$= - \tilde{\Pi}_1^{(P)}(\rho,q_{12})\delta_{1M} + \tilde{\Pi}_2^{(P)}(\rho,q_{22})\delta_{2M} \ , \tag{2.53}$$

$$A_1k_1'(q_{12}) + B_1k_2'(q_{12}) - A_2\left(\left|\frac{\alpha_1}{\alpha_2}\right|\right)^{2/3}\frac{\alpha_2}{\alpha_1}k_1'(q_{22}) - B_2\left(\left|\frac{\alpha_1}{\alpha_2}\right|\right)^{2/3}\frac{\alpha_2}{\alpha_1}k_2'(q_{22})$$

$$= - \frac{1}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\frac{d}{dz}\left\{\tilde{\Pi}_1^{(P)}(\rho,q_{12})\right\}\delta_{1M}$$

$$+ \frac{1}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\frac{d}{dz}\left\{\tilde{\Pi}_2^{(P)}(\rho,q_{22})\right\}\delta_{2M} \ , \tag{2.54}$$

$$A_2k_1(q_{23}) + B_2k_2(q_{23}) - B_3h_2(q_{33})$$

$$= -\tilde{\Pi}_2^{(P)}(\rho,q_{23})\delta_{2M} + \tilde{\Pi}_3^{(P)}(\rho,q_{33})\delta_{3M} \ , \tag{2.55}$$

and

$$A_2k_1'(q_{23}) + B_2k_2'(q_{23}) - B_3\left(\left|\frac{\alpha_2}{\alpha_3}\right|\right)^{2/3}\frac{\alpha_3}{\alpha_2}h_2'(q_{33})$$

$$= - \frac{1}{\alpha_2}\left(\frac{|\alpha_2|}{k}\right)^{2/3}\frac{d}{dz}\left\{\tilde{\Pi}_2^{(P)}(\rho,q_{23})\right\}\delta_{2M}$$

$$+ \frac{1}{\alpha_2} \left(\frac{|\alpha_2|}{k}\right)^{2/3} \frac{d}{dz} \left\{\tilde{\Pi}_3^{(P)}(\rho, q_{33})\right\} \delta_{3M} \ , \tag{2.56}$$

where

$$\hat{k}_1(q_{11}) = k_1'(q_{11}) - G_{SH} k_1(q_{11}) \tag{2.57}$$

and

$$\hat{k}_2(q_{11}) = k_2'(q_{11}) - G_{SH} k_2(q_{11}) \tag{2.58}$$

with

$$G_{SH} = \frac{i\gamma}{\alpha_1} \left(\frac{|\alpha_1|}{k}\right)^{2/3}$$

$$= i\left(\frac{k}{\alpha_1}\right)^{1/3} \sqrt{n_g^2 - \beta^2} \qquad (\alpha_1 > 0) \ . \tag{2.59}$$

The notation $q_{j\ell}$ means

$$q_{j\ell} = q_j(z_\ell) \tag{2.60}$$

and

$$q_{jT} = q_j(z_T) \ . \tag{2.61}$$

in equations (2.52-2.56) the prime denotes the derivative with respect to the argument.

The derivatives $\frac{d}{dz}\left\{\tilde{\Pi}_j^{(P)}(\rho, q_{j\ell})\right\}$ in equations (2.52-2.56) are equal to

$$\frac{d}{dz}\left\{\tilde{\Pi}_j^{(P)}(\rho, q_{jj})\right\} = \left(\frac{k}{|\alpha_j|}\right)^{2/3} \alpha_j R_j k_1'(q_{jj}) k_2(q_{jT}) \qquad (j = 1,2) \ , \tag{2.62}$$

$$\frac{d}{dz}\left\{\tilde{\Pi}_3^{(P)}(\rho, q_{33})\right\} = \left(\frac{k}{|\alpha_3|}\right)^{2/3} \alpha_3 R_3 \left[h_1'(q_{33}) - e^{-4\pi i/3} h_2'(q_{33})\right] h_2(q_{3T}) \ , \tag{2.63}$$

$$\frac{d}{dz}\left\{\tilde{\Pi}_j^{(P)}(\rho,q_{j,j+1})\right\} = \left(\frac{k}{|\alpha_j|}\right)^{2/3}\alpha_j R_j k_1(q_{jT})k_2(q_{j,j+1}) \qquad (j = 1,2)\ , \quad (2.64)$$

or

$$\frac{d}{dz}\left\{\tilde{\Pi}_j^{(P)}(\rho,q_{jj})\right\} = -\left(\frac{k}{|\alpha_j|}\right)^{2/3}\alpha_j R_j k_1(q_{jT})k_2'(q_{jj}) \qquad (j = 1,2)\ , \quad (2.65)$$

$$\frac{d}{dz}\left\{\tilde{\Pi}_3^{(P)}(\rho,q_{33})\right\} = -\left(\frac{k}{|\alpha_3|}\right)^{2/3}\alpha_3 R_3\left[h_1(q_{3T})-e^{-4\pi i/3}h_2(q_{3T})\right]h_2'(q_{33})\ , \quad (2.66)$$

$$\frac{d}{dz}\left\{\tilde{\Pi}_j^{(P)}(\rho,q_{j,j+1})\right\} = -\left(\frac{k}{|\alpha_j|}\right)^{2/3}\alpha_j R_j k_1'(q_{j,j+1})k_2(q_{jT}) \qquad (j = 1,2)\ , \quad (2.67)$$

depending upon the choice of particular solution (2.40, 2.41).

In matrix notation, equations (2.52-2.56) may be written as

$$\underset{\sim}{\Delta}\bar{\zeta} = \bar{\beta}\ , \quad (2.68)$$

where the coefficient matrix $\underset{\sim}{\Delta}$ is given by

$$\underset{\sim}{\Delta} = \begin{pmatrix} \Delta_{11} & \Delta_{12} & 0 & 0 & 0 \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & \Delta_{24} & 0 \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & \Delta_{34} & 0 \\ 0 & 0 & \Delta_{43} & \Delta_{44} & \Delta_{45} \\ 0 & 0 & \Delta_{53} & \Delta_{54} & \Delta_{55} \end{pmatrix} \quad (2.69)$$

and the vectors $\bar{\zeta}$ and $\bar{\beta}$ are equal to

$$\bar{\zeta} = \begin{pmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ B_3 \end{pmatrix} \quad (2.70)$$

and

$$\bar{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix} \tag{2.71}$$

The non-zero elements of the 5×5 coefficient matrix $\underset{\sim}{\Delta}$ are

$$\Delta_{11} = \hat{k}_1(q_{11}) \ , \tag{2.72}$$

$$\Delta_{12} = \hat{k}_2(q_{11}) \ , \tag{2.73}$$

$$\Delta_{21} = k_1(q_{12}) \ , \tag{2.74}$$

$$\Delta_{22} = k_2(q_{12}) \ , \tag{2.75}$$

$$\Delta_{23} = -k_1(q_{22}) \ , \tag{2.76}$$

$$\Delta_{24} = -k_2(q_{22}) \ , \tag{2.77}$$

$$\Delta_{31} = k_1'(q_{12}) \ , \tag{2.78}$$

$$\Delta_{32} = k_2'(q_{12}) \ , \tag{2.79}$$

$$\Delta_{33} = -\left(\left|\frac{\alpha_1}{\alpha_2}\right|\right)^{2/3}\frac{\alpha_2}{\alpha_1}k_1'(q_{22}) \ , \tag{2.80}$$

$$\Delta_{34} = -\left(\left|\frac{\alpha_1}{\alpha_2}\right|\right)^{2/3}\frac{\alpha_2}{\alpha_1}k_2'(q_{22}) \ , \tag{2.81}$$

$$\Delta_{43} = k_1(q_{23}) \ , \tag{2.82}$$

$$\Delta_{44} = k_2(q_{23}) \ , \tag{2.83}$$

$$\Delta_{45} = -h_2(q_{33}) \ , \tag{2.84}$$

$$\Delta_{53} = k_1'(q_{23}) \ , \tag{2.85}$$

$$\Delta_{54} = k_2'(q_{23}) \ , \tag{2.86}$$

and

$$\Delta_{55} = -\left(\left|\frac{\alpha_2}{\alpha_3}\right|\right)^{2/3}\frac{\alpha_3}{\alpha_2}h_2'(q_{33}) \ . \tag{2.87}$$

The elements of the vector $\bar{\beta}$ are

$$\beta_1 = \frac{i\gamma}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\tilde{\Pi}_1^{(P)}(\rho,q_{11})\delta_{1M}$$

$$- \frac{1}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\frac{d}{dz}\left\{\tilde{\Pi}_1^{(P)}(\rho,q_{11})\right\}\delta_{1M} \ , \tag{2.88}$$

$$\beta_2 = -\tilde{\Pi}_1^{(P)}(\rho,q_{12})\delta_{1M} + \tilde{\Pi}_2^{(P)}(\rho,q_{22})\delta_{2M} \ , \tag{2.89}$$

$$\beta_3 = -\frac{1}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\frac{d}{dz}\left\{\tilde{\Pi}_1^{(P)}(\rho,q_{12})\right\}\delta_{1M}$$

$$+ \frac{1}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\frac{d}{dz}\left\{\tilde{\Pi}_2^{(P)}(\rho,q_{22})\right\}\delta_{2M} \ , \tag{2.90}$$

$$\beta_4 = -\tilde{\Pi}_2^{(P)}(\rho,q_{23})\delta_{2M} + \tilde{\Pi}_3^{(P)}(\rho,q_{23})\delta_{3M} \ , \tag{2.91}$$

and

$$\beta_5 = -\frac{1}{\alpha_2}\left(\frac{|\alpha_2|}{k}\right)^{2/3}\frac{d}{dz}\left\{\tilde{\Pi}_2^{(P)}(\rho,q_{23})\right\}\delta_{2M}$$

$$+ \frac{1}{\alpha_2}\left(\frac{|\alpha_2|}{k}\right)^{2/3}\frac{d}{dz}\left\{\tilde{\Pi}_3^{(P)}(\rho,q_{33})\right\}\delta_{3M} \ . \tag{2.92}$$

The solution for the unknown coefficients A_j and B_j may be obtained from equation (2.68) by using Cramer's rule [14]. Thus

$$A_j = \frac{||\underset{\sim}{\tau}_{A_j}||}{||\underset{\sim}{\Delta}||} \tag{2.93}$$

and

$$B_j = \frac{||\underset{\sim}{\tau}_{B_j}||}{||\underset{\sim}{\Delta}||} \ , \tag{2.94}$$

where for $j = 1,2$, the matrix $\underset{\sim}{\Delta}_{Aj}$ is obtained by replacing the column of $\underset{\sim}{\Delta}$ in equation (2.69) containing the coefficients of A_j with the vector $\bar{\beta}$; and for $j = 3$, $||\underset{\sim}{\Delta}_{A3}|| = 0$. The matrix $\underset{\sim}{\Delta}_{Bj}$ is obtained by replacing the column of $\underset{\sim}{\Delta}$ containing the coefficients of B_j with the vector $\bar{\beta}$. The notation $||\underset{\sim}{\Delta}||$ indicates the determinant of the matrix $\underset{\sim}{\Delta}$.

Substituting equations (2.93, 2.94) into equations (2.48, 2.49) yields

$$\tilde{\Pi}_j(\rho,z) = \frac{1}{||\underset{\sim}{\Delta}||}\{||\underset{\sim}{\Delta}_{Aj}||k_1(q_j) + ||\underset{\sim}{\Delta}_{Bj}||k_2(q_j)\}$$

$$+ \tilde{\Pi}_M^{(P)}(\rho,q_M)\delta_{jM} \qquad (j = 1,2) \qquad (2.95)$$

and

$$\tilde{\Pi}_3(\rho,z) = \frac{1}{||\underset{\sim}{\Delta}||}\{||\underset{\sim}{\Delta}_{B3}||h_2(q_3)\} + \tilde{\Pi}_M^{(P)}(\rho,q_M)\delta_{3M} \ . \qquad (2.96)$$

From equation (2.19), the Hertz magnetic potential $\Pi_j(r,z)$ can be expressed as the contour integral

$$\Pi_j(r,z) = \frac{1}{4\pi} \int_C \rho d\rho H_o^{(2)}(\rho r)\tilde{\Pi}_j(\rho,z) \ , \qquad (2.97)$$

where the contour C is shown in figure 2. By closing the contour C and using the theory of residues, equation (2.97) becomes

$$\Pi_j(r,z) = -\frac{2\pi i}{4\pi} \sum_n b_j(n) + \frac{1}{4\pi} \int_{C_1} \rho d\rho H_o^{(2)}(\rho r)\tilde{\Pi}_j(\rho,z)$$

$$+ \frac{1}{4\pi} \int_B \rho d\rho H_o^{(2)}(\rho r)\tilde{\Pi}_j(\rho,z) \ , \qquad (2.98)$$

where the $b_j(n)$ are the residues of $\rho H_o^{(2)}(\rho r)\tilde{\Pi}_j(\rho,z)$ at the poles of $\tilde{\Pi}_j(\rho,z)$ in the fourth quadrant; the contour C_1 is the quarter-circle of infinite radius lying in the fourth quadrant; and the contour B encloses the branch cut which is present as a result of the square root in the expression for γ in equation (2.27). Through use of the asymptotic approximations for the Hankel and Airy functions it can be shown that

$$\frac{1}{4\pi} \int_{C_1} \rho d\rho H_o^{(2)}(\rho r)\tilde{\Pi}_j(\rho,z) = 0 \ . \qquad (2.99)$$

The integral over the contour B represents the surface wave contribution to the total field and is assumed to be small relative to the other field contributions. Therefore,

$$\Pi_j(r,z) \cong -\frac{i}{2} \sum_n b_j(n) \ . \tag{2.100}$$

The residues $b_j(n)$ are given by [2]

$$b_j(n) = \rho_n H_o^{(2)}(\rho_n r) \frac{\left\{ ||\underset{\sim}{I}_{Aj}||k_1(q_j) + ||\underset{\sim}{I}_{Bj}||k_2(q_j) \right\}_{\rho=\rho_n}}{\left\{ \frac{\partial}{\partial\rho} ||\underset{\sim}{\Delta}|| \right\}_{\rho=\rho_n}} \quad (j = 1,2) \tag{2.101}$$

and

$$b_3(n) = \rho_n H_o^{(2)}(\rho_n r) \frac{\left\{ ||\underset{\sim}{I}_{B3}||h_2(q_3) \right\}_{\rho=\rho_n}}{\left\{ \frac{\partial}{\partial\rho} ||\underset{\sim}{\Delta}|| \right\}_{\rho=\rho_n}} \ , \tag{2.102}$$

where ρ_n is the n-th value of ρ for which $||\underset{\sim}{\Delta}|| = 0$.

Substituting equations (2.101, 2.102) into (2.100) yields

$$\Pi_j(r,z) = -\frac{i}{2} \sum_n \lambda_n E(n,z,z_T) \sqrt{\rho_n}\, H_o^{(2)}(\rho_n r) \ , \tag{2.103}$$

where

$$\lambda_n = \frac{\sqrt{\rho_n}}{\left\{ \frac{\partial}{\partial\rho} ||\underset{\sim}{\Delta}|| \right\}_{\rho=\rho_n}}$$

$$= \frac{k^2}{2\sqrt{\rho_n} \left(\frac{k}{|\alpha_1|} \right)^{2/3} \left\{ \frac{\partial}{\partial q_{11}} ||\underset{\sim}{\Delta}|| \right\}_{q_{11}=q_{11}(\rho_n)}} \tag{2.104}$$

is the modal excitation factor and

$$E(n,z,z_T) = \begin{cases} \{||\underset{\sim}{\tau}_{Aj}||k_1(q_j) + ||\underset{\sim}{\tau}_{Bj}||k_2(q_j)\}_{\rho=\rho_n} & (j = 1,2) \\ \{||\underset{\sim}{\tau}_{B3}||h_2(q_3)\}_{\rho=\rho_n} & (j = 3) \end{cases} \tag{2.105}$$

is the modal height-gain function.

For the cases of interest $|\rho_n r| >> 1$, so that the asymptotic approximation [15]

$$H_o^{(2)}(\rho_n r) \sim \left(\frac{2}{\pi\rho_n r}\right)^{1/2} \exp(-i\rho_n r + i\frac{\pi}{4}) \tag{2.106}$$

can be used in equation (2.103). This yields

$$\Pi_j(r,z) = -\frac{i}{\sqrt{2\pi r}} e^{i\pi/4} \sum_n \lambda_n E(n,z,z_T) e^{-i\rho_n r} . \tag{2.107}$$

Rather than evaluate the fields by using equation (2.107) together with equations (2.17, 2.18), it is more convenient to evaluate the magnitude of the electric field relative to its free space value. This is given by [2]

$$A_{coh} = \frac{|\bar{E}|}{|\bar{E}_{fs}|} = \frac{2\sqrt{2\pi r}}{p} \left|\sum_n \lambda_n E(n,z,z_T) e^{-i\rho_n r}\right| , \tag{2.108}$$

where p is the magnetic dipole strength. This expression takes into account the phase of each term in the sum and is known as the coherent mode sum. It is also useful to define the relative field strength that results when the phase of each term in (2.107) is ignored. This is the incoherent mode sum [2]

$$A_{incoh} = \frac{2\sqrt{2\pi r}}{p} \left[\sum_n |\lambda_n E(n,z,z_T) e^{-i\rho_n r}|^2\right]^{1/2} . \tag{2.109}$$

Because $E(n,z,z_T)$ is proportional to $R_M = \frac{p}{Wq_M'}$ through the coefficients A_j and B_j, the factor of p in $E(n,z,z_T)$ will cancel the factor of p^{-1} in the expressions (2.108, 2.109) for A_{coh} and A_{incoh}.

VERTICALLY POLARIZED WAVE PROPAGATION

Vertically polarized waves may be considered to be due to a radiating electric dipole $\bar{p}_e$ oriented in the z-direction and located at $x = y = 0$, $z = z_T$. A time dependence of $e^{+i\omega t}$ again will be assumed. The medium again is assumed to be laterally homogeneous so that the fields due to the dipole exhibit axial symmetry. Because of the variation of the permittivity ε with height above the surface of the earth, the equations

$$\bar{H}(r,z) = i\varepsilon\omega\bar{\nabla} \times \bar{\Pi}^*(r,z) \tag{2.110}$$

and

$$\bar{E}(r,z) = \bar{\nabla} \times \bar{\nabla} \times \bar{\Pi}^*(r,z) \quad , \tag{2.111}$$

relating the electric and magnetic fields to the electric Hertz potential vector

$$\bar{\Pi}^*(r,z) = \Pi^*(r,z)\bar{e}_z \quad , \tag{2.112}$$

are only approximately consistent with Maxwell's equations. However, because the variation of ε_r with height z in the atmosphere is so small, equations (2.110, 2.111) constitute a good approximation [11].

The calculation of the electric Hertz potential $\Pi^*(r,z)$ proceeds exactly as the calculation of the magnetic Hertz potential $\Pi(r,z)$, with some minor modifications. The magnetic dipole strength p in the differential equation (2.20) should be replaced by

$$p^* = \frac{p_e}{\varepsilon} \quad . \tag{2.113}$$

This yields the differential equation

$$\left\{\frac{d^2}{dz^2} + k^2\left[m_j^2(z) - \frac{\rho^2}{k^2}\right]\right\} \tilde{\Pi}_j^*(\rho,z) = -p^*\delta(z - z_T) \quad . \tag{2.114}$$

The continuity conditions (2.22-2.25) become

$$n_g^2 \tilde{\Pi}_g^*(\rho,o) = n_1^2(o)\tilde{\Pi}_1^*(\rho,o) \ , \tag{2.115}$$

$$\frac{d}{dz}\{\tilde{\Pi}_g^*(\rho,o)\} = \frac{d}{dz}\{\tilde{\Pi}_1^*(\rho,o)\} \ , \tag{2.116}$$

$$\tilde{\Pi}_j^*(\rho,z_{j+1}) = \tilde{\Pi}_{j+1}^*(\rho,z_{j+1}) \qquad (j = 1,2) \ , \tag{2.117}$$

and

$$\frac{d}{dz}\{\tilde{\Pi}_j^*(\rho,z_{j+1})\} = \frac{d}{dz}\{\tilde{\Pi}_j^*(\rho,z_{j+1})\} \qquad (j = 1,2) \ . \tag{2.118}$$

The particular solutions $\tilde{\Pi}_M^{*(P)}(\rho,z)$ of (2.114) are the same as the particular solutions $\tilde{\Pi}_M^{(P)}(\rho,z)$ given by equations (2.40, 2.41) except that R_M is replaced by

$$R_M^* = \frac{p^*}{Wq_M'} \ . \tag{2.119}$$

The system of equations (2.68) becomes

$$\underset{\sim}{\Delta}^*\tilde{\zeta}^* = \bar{\beta}^* \ . \tag{2.120}$$

The elements of the coefficient matrix $\underset{\sim}{\Delta}^*$ are identical to the elements of $\underset{\sim}{\Delta}$ in equations (2.72-2.87) except that $\hat{k}_1(q_{11})$ and $\hat{k}_2(q_{11})$, as given by equations (2.57, 2.58), are now equal to

$$\hat{k}_1(q_{11}) = k_1'(q_{11}) - G_{SV}k_1(q_{11}) \tag{2.121}$$

and

$$\hat{k}_2(q_{11}) = k_2'(q_{11}) - G_{SV}k_2(q_{11}) \ , \tag{2.122}$$

where

$$G_{SV} = \frac{i\gamma}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\frac{n_f^2(0)}{n_g^2} \ . \tag{2.123}$$

This change is due to the continuity condition (2.115). When considering the field strength relative to its free space value, the dipole strength p* cancels out in the final results for A_{coh}^* and A_{incoh}^*, just as p does in A_{coh} and A_{incoh}. Therefore, the only significant change for vertical polarization

is the replacement of G_{SH} by G_{SV} in the expressions for $\hat{k}_1(q_{11})$ and $\hat{k}_2(q_{11})$, due to the changed continuity condition (2.115).

TRILINEAR DUCT MODAL FUNCTION

When evaluating the field strength relative to its free space value, equations (2.108, 2.109), the values $\rho = \rho_n$ for which $||\underset{\sim}{\Delta}|| = 0$ are needed. The determinant $||\underset{\sim}{\Delta}||$ is known as the modal function. The values $\rho = \rho_n$ for which $||\underset{\sim}{\Delta}|| = 0$ are the eigenvalues of the problem, each eigenvalue representing a different electromagnetic mode that propagates in the atmospheric waveguide formed by the trilinear refractivity profile. In XWVG the zeros of $||\underset{\sim}{\Delta}||$ are found by using the complex root-finding routine of Shellman and Morfitt, which will be discussed in a later section of this report. First, explicit formulas will be given for the modal function, its derivative, and the modal height-gain functions for the case of a trilinear refractivity profile.

The determinant of the coefficient matrix $\underset{\sim}{\Delta}$, equation (2.69), can be expanded to obtain

$$||\underset{\sim}{\Delta}|| = \mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} \left\{ \mp \left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} \zeta(q_{11})\chi(q_{11}) + \psi(q_{11})\phi(q_{11}) \right\} , \quad (2.124)$$

where both α_1 and α_3 are positive and the top sign is to be taken if $\alpha_2 > 0$ and the bottom sign if $\alpha_2 < 0$. The auxiliary functions $\zeta(q_{11})$, $\chi(q_{11})$, $\psi(q_{11})$, and $\phi(q_{11})$ are given by

$$\zeta(q_{11}) = k_2(q_{12})\hat{k}_1(q_{11}) - k_1(q_{12})\hat{k}_2(q_{11}) , \quad (2.125)$$

$$\chi(q_{11}) = h_2'(q_{33})[k_1'(q_{22})k_2(q_{23}) - k_2'(q_{22})k_1(q_{23})] \mp \left(\frac{|\alpha_2|}{\alpha_3}\right)^{1/3} h_2(q_{33})[k_1'(q_{22})k_2'(q_{23}) - k_2'(q_{22})k_1'(q_{23})] , \quad (2.126)$$

$$\psi(q_{11}) = k_2'(q_{12})\hat{k}_1(q_{11}) - k_1'(q_{12})\hat{k}_2(q_{11}) , \quad (2.127)$$

and

$$\phi(q_{11}) = h_2'(q_{33})[k_1(q_{22})k_2(q_{23}) - k_2(q_{22})k_1(q_{23})] \mp \left(\frac{|\alpha_2|}{\alpha_3}\right)^{1/3} h_2(q_{33})[k_1(q_{22})k_2'(q_{23}) - k_2(q_{22})k_1'(q_{23})] . \quad (2.128)$$

The arguments q_{ij} appearing in equations (2.125-2.128) are equal to

$$q_{11} = \left(\frac{k}{\alpha_1}\right)^{2/3} [1 + i\eta - \alpha_1 H - \beta^2] \ , \tag{2.129}$$

$$q_{12} = q_{11} + \left(\frac{k}{\alpha_1}\right)^{2/3} \alpha_1 z_2 \ , \tag{2.130}$$

$$q_{22} = \left(\frac{\alpha_1}{|\alpha_2|}\right)^{2/3} q_{11} + \left(\frac{k}{|\alpha_2|}\right)^{2/3} \alpha_1 z_2 \ , \tag{2.131}$$

$$q_{23} = \left(\frac{\alpha_1}{|\alpha_2|}\right)^{2/3} q_{11} + \left(\frac{k}{|\alpha_2|}\right)^{2/3} [\alpha_1 z_2 + \alpha_2 (z_3 - z_2)] \ , \tag{2.132}$$

and

$$q_{33} = \left(\frac{\alpha_1}{\alpha_3}\right)^{2/3} q_{11} + \left(\frac{k}{\alpha_3}\right)^{2/3} [\alpha_1 z_2 + \alpha_2 (z_3 - z_2)] \ , \tag{2.133}$$

with

$$\beta^2 = \frac{\rho^2}{k^2} \ . \tag{2.134}$$

The functions $\hat{k}_1(q_{11})$ and $\hat{k}_2(q_{11})$ are given by

$$\hat{k}_1(q_{11}) = k_1'(q_{11}) - G_S k_1(q_{11}) \tag{2.135}$$

and

$$\hat{k}_2(q_{11}) = k_2'(q_{11}) - G_S k_2(q_{11}) \ , \tag{2.136}$$

where

$$G_S = G_{SH} = \frac{ik}{\alpha_1} \left(\frac{|\alpha_1|}{k}\right)^{2/3} \sqrt{n_g^2 - \beta^2} \tag{2.137}$$

for horizontal polarization and

$$G_S = G_{SV} = \frac{n_1^2(0)}{n_g^2} G_{SH} \tag{2.138}$$

for vertical polarization.

The derivative of the modal function $||\underset{\sim}{\Delta}||$ with respect to q_{11} is needed by the Shellman and Morfitt root-finding routine. The derivative is also needed for the evaluation of the modal excitation factors λ_n. Taking the derivative of equation (2.124) and making use of Stokes' differential equation to eliminate the second derivatives that occur yields

$$\frac{\partial}{\partial q_{11}}\{||\underset{\sim}{\Delta}||\} = \mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3}\left\{ \mp \left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3}\left[\zeta'(q_{11})\chi(q_{11}) + \zeta(q_{11})\chi'(q_{11})\right]\right.$$

$$\left. + [\psi'(q_{11})\phi(q_{11}) + \psi(q_{11})\phi'(q_{11})]\right\}, \qquad (2.139)$$

where again the upper sign is to be taken when $\alpha_2 > 0$ and the lower sign when $\alpha_2 < 0$. The derivatives with respect to q_{11} of the auxiliary functions are

$$\zeta'(q_{11}) = k_2'(q_{12})k_1(q_{11}) - k_1'(q_{12})k_2(q_{11}) + k_1(q_{12})\left\{\left[q_{11} + \frac{\partial G_S}{\partial q_{11}}\right]k_2(q_{11})\right.$$

$$\left. + G_S k_2'(q_{11})\right\} - k_2(q_{12})\left\{\left[q_{11} + \frac{\partial G_S}{\partial q_{11}}\right]k_1(q_{11}) + G_S k_1'(q_{11})\right\}, \qquad (2.140)$$

$$\chi'(q_{11}) = -q_{33}\left(\frac{\alpha_1}{\alpha_3}\right)^{2/3} h_2(q_{33})[k_1'(q_{22})k_2(q_{23}) - k_2'(q_{22})k_1(q_{23})]$$

$$+ \left(\frac{\alpha_1}{|\alpha_2|}\right)^{2/3} h_2'(q_{33})\{q_{22}[k_2(q_{22})k_1(q_{23}) - k_1(q_{22})k_2(q_{23})]$$

$$+ [k_1'(q_{22})k_2'(q_{23}) - k_2'(q_{22})k_1'(q_{23})]\}$$

$$\mp \left(\frac{|\alpha_2|}{\alpha_3}\right)^{1/3}\left(\frac{\alpha_1}{\alpha_3}\right)^{2/3} h_2'(q_{33})[k_1'(q_{22})k_2'(q_{23}) - k_2'(q_{22})k_1'(q_{23})]$$

$$\mp \left(\frac{|\alpha_2|}{\alpha_3}\right)^{1/3}\left(\frac{\alpha_1}{|\alpha_2|}\right)^{2/3} h_2(q_{33})\{q_{22}[k_2(q_{22})k_1'(q_{23})$$

$$- k_1(q_{22})k_2'(q_{23})] + q_{23}[k_2'(q_{22})k_1(q_{23}) - k_1'(q_{22})k_2(q_{23})]\}, \qquad (2.141)$$

$$\psi'(q_{11}) = - q_{12}k_2(q_{12})\hat{k}_1(q_{11}) - k_2'(q_{12})\left\{\left[q_{11} + \frac{\partial G_S}{\partial q_{11}}\right] k_1(q_{11}) + G_S k_1'(q_{11})\right\}$$

$$+ q_{12}k_1(q_{12})\hat{k}_2(q_{11}) + k_1'(q_{12})\left\{\left[q_{11} + \frac{\partial G_S}{\partial q_{11}}\right] k_2(q_{11}) + G_S k_2'(q_{11})\right\}, \quad (2.142)$$

and

$$\phi'(q_{11}) = -q_{33}\left(\frac{\alpha_1}{\alpha_3}\right)^{2/3} h_2(q_{33})[k_1(q_{22})k_2(q_{23}) - k_2(q_{22})k_1(q_{23})]$$

$$+\left(\frac{\alpha_1}{|\alpha_2|}\right)^{2/3} h_2'(q_{33})\{[k_1'(q_{22})k_2(q_{23}) - k_2'(q_{22})k_1(q_{23})]$$

$$+ [k_1(q_{22})k_2'(q_{23}) - k_2(q_{22})k_1'(q_{23})]\}$$

$$\mp \left(\frac{|\alpha_2|}{\alpha_3}\right)^{1/3}\left(\frac{\alpha_1}{\alpha_3}\right)^{2/3} h_2'(q_{33})[k_1(q_{22})k_2'(q_{23}) - k_2(q_{22})k_1'(q_{23})]$$

$$\mp \left(\frac{|\alpha_2|}{\alpha_3}\right)^{1/3}\left(\frac{\alpha_1}{|\alpha_2|}\right)^{2/3} h_2(q_{33})\{[k_1'(q_{22})k_2'(q_{23}) - k_2'(q_{22})k_1'(q_{23})]$$

$$- q_{23}[k_1(q_{22})k_2(q_{23}) - k_2(q_{22})k_1(q_{23})]\} . \quad (2.143)$$

For horizontal polarization

$$\frac{\partial G_S}{\partial q_{11}} = \frac{\partial G_{SH}}{\partial q_{11}} = \frac{i}{2}\left(\frac{\alpha_1}{k}\right)^{1/3}\left[n_g^2 - \beta^2\right]^{-1/2} , \quad (2.144)$$

while for vertical polarization

$$\frac{\partial G_S}{\partial q_{11}} = \frac{\partial G_{SV}}{\partial q_{11}} = \frac{n_1^2(0)}{n_g^2}\frac{\partial G_{SH}}{\partial q_{11}} . \quad (2.145)$$

TRILINEAR DUCT MODAL HEIGHT-GAIN FUNCTIONS

The height-gain function corresponding to the n-th mode is

$$E(n,z,z_T) = \begin{cases} \{||\underset{\sim}{\tau}_{Aj}||k_1(q_j) + ||\underset{\sim}{\tau}_{Bj}||k_2(q_j)\}_{\rho=\rho_n} & (j = 1,2) \\ \\ \{||\underset{\sim}{\tau}_{B3}||h_2(q_3)\}_{\rho=\rho_n} & (j = 3) \ . \end{cases} \tag{2.146}$$

When $j = 1,2$, the matrix $\underset{\sim}{\tau}_{Aj}$ is the matrix obtained by replacing the column of the the matrix $\underset{\sim}{\Delta}$, equation (2.69), containing the coefficients of A_j by the vector $\bar{\beta}$; and when $j = 3$, $||\underset{\sim}{\tau}_{A3}|| = 0$. The matrix $\underset{\sim}{\tau}_{Bj}$ is obtained by replacing the column of $\underset{\sim}{\Delta}$ containing the coefficients of B_j by the vector $\bar{\beta}$. For example, $\underset{\sim}{\tau}_{B2}$ is given by

$$\underset{\sim}{\tau}_{B2} = \begin{pmatrix} \Delta_{11} & \Delta_{12} & 0 & \beta_1 & 0 \\ \Delta_{21} & \Delta_{22} & \Delta_{23} & \beta_2 & 0 \\ \Delta_{31} & \Delta_{32} & \Delta_{33} & \beta_3 & 0 \\ 0 & 0 & \Delta_{43} & \beta_4 & \Delta_{45} \\ 0 & 0 & \Delta_{53} & \beta_5 & \Delta_{55} \end{pmatrix} \tag{2.147}$$

The elements of the matrix $\underset{\sim}{\Delta}$ are given by equations (2.72-2.87) and the elements of the vector $\bar{\beta}$ are given by equations (2.88-2.92).

The elements of the vector $\bar{\beta}$ depend upon the choice of particular solution. Theoretically, either of the particular solutions

$$\tilde{\Pi}_M^{(P)}(\rho,q_M) = \begin{cases} R_M k_1(q_{M<})k_2(q_{M>}) & (M = 1,2) \\ \\ R_M\left[h_1(q_{M<})-e^{-4\pi i/3}h_1(q_{M<})\right]h_2(q_{M>}) & (M = 3) \end{cases} \tag{2.148}$$

or

$$\tilde{\Pi}_M^{(P)}(\rho,q_M) = \begin{cases} -R_M k_1(q_{M>})k_2(q_{M<}) & (M = 1,2) \\ \\ -R_M\left[h_1(q_{M>})-e^{-4\pi i/3}h_1(q_{M>})\right]h_2(q_{M<}) & (M = 3) \end{cases} \tag{2.149}$$

can be used. Numerically, however, one solution may be preferable to the other. Numerical experience shows that for the trilinear refractivity profiles shown in figure 1, when the transmitter and receiver are both in the first layer or the transmitter is in the second layer and $\frac{dM_2}{dz}$ is negative, the best choice for the particular solution is (2.149). When the transmitter is in the first layer and the receiver is in the second or third layer, when the transmitter is in the second layer and $\frac{dM_2}{dz}$ is positive, or when the transmitter is in the third layer, the best choice for the particular solution is (2.148).

When the transmitter is in the first layer and the receiver is in the first layer, the modal height-gain function is given by

$$
\begin{aligned}
E(n,z_R,z_T) = R_1 \Bigg\{ & \Bigg[\Bigg\{ \mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33}) k_2(q_{23}) \\
& + h_2(q_{33}) k_2'(q_{23}) \Bigg\} \Bigg\{ \mp \left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k_1'(q_{22}) k_2(q_{12}) + k_1(q_{22}) k_2'(q_{12}) \Bigg\} \\
& + \Bigg\{ \mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33}) k_1(q_{23}) + h_2(q_{33}) k_1'(q_{23}) \Bigg\} \Bigg\{ \pm \left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k_2'(q_{22}) k_2(q_{12}) \\
& - k_2(q_{22}) k_2'(q_{12}) \Bigg\} \Bigg] \hat{k}_2(q_{11}) k_1(q_{1T}) k_1(q_{1R}) + \Bigg[\Bigg\{ \mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33}) k_2(q_{23}) \\
& + h_2(q_{33}) k_2'(q_{23}) \Bigg\} \Bigg\{ \pm \left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k_1'(q_{22}) k_1(q_{12}) - k_1(q_{22}) k_1'(q_{12}) \Bigg\} \\
& + \Bigg\{ \mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33}) k_1(q_{23}) + h_2(q_{33}) k_1'(q_{23}) \Bigg\} \Bigg\{ \mp \left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k_2'(q_{22}) k_1(q_{12}) \\
& + k_2(q_{22}) k_1'(q_{12}) \Bigg\} \Bigg] \hat{k}_2(q_{11}) \{ k_2(q_{1T}) k_1(q_{1R}) + k_1(q_{1T}) k_2(q_{1R}) \}
\end{aligned}
$$

$$+ \left[\left\{\mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33})k_2(q_{23}) + h_2(q_{33})k_2'(q_{23})\right\}\left\{\mp \left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k_1'(q_{22})k_1(q_{12}) + k_1(q_{22})k_1'(q_{12})\right\}\right.$$

$$\left. + \left\{\mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33})k_1(q_{23}) + h_2(q_{33})k_1'(q_{23})\right\}\left\{\pm \left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k_2'(q_{22})k_1(q_{12}) - k_2(q_{22})k_1'(q_{12})\right\}\right] \hat{k}_1(q_{11})k_2(q_{1T})k_2(q_{1R}) \Bigg\}$$

$$(0 \le z_R \le z_2 \; ; \; 0 \le z_T \le z_2) \; . \qquad (2.150)$$

When the transmitter is in the first layer and the receiver is in the second layer, the modal height-gain function is given by

$$E(n,z_R,z_T) = R_1 W \Bigg\{ \left[\mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33})k_2(q_{23}) + h_2(q_{33})k_2'(q_{23})\right] \hat{k}_2(q_{11})k_1(q_{1T})k_1(q_{2R})$$

$$- \left[\mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33})k_2(q_{23}) + h_2(q_{33})k_2'(q_{23})\right] \hat{k}_1(q_{11})k_2(q_{1T})k_1(q_{2R})$$

$$- \left[\mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33})k_1(q_{23}) + h_2(q_{33})k_1'(q_{23})\right] \hat{k}_2(q_{11})k_1(q_{1T})k_2(q_{2R})$$

$$+ \left[\mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33})k_1(q_{23}) + h_2(q_{33})k_1'(q_{23})\right] \hat{k}_1(q_{11})k_2(q_{1T})k_2(q_{2R}) \Bigg\}$$

$$(z_2 \le z_R \le z_3 \; ; \; 0 \le z_T \le z_2) \; . \qquad (2.151)$$

When the transmitter is in the first layer and the receiver is in the third layer, the modal height-gain function is

$$E(n,z_R,z_T) = -R_1 W^2 [\hat{k}_1(q_{11})k_2(q_{1T}) - \hat{k}_2(q_{11})k_1(q_{1T})]k_2(q_{3R})$$

$$(z_3 \le z_R \; ; \; 0 \le z_T \le z_2) \; . \qquad (2.152)$$

In the above equations

$$R_1 = \frac{p}{W q_1'} , \tag{2.153}$$

where p is the magnetic dipole strength, W is the Wronskian of $k_1(\xi)$ and $k_2(\xi)$ and has the value

$$W = -\frac{4i}{\pi}\left(\frac{3}{2}\right)^{1/3} , \tag{2.154}$$

and q_1' is equal to

$$q_1' = \frac{\partial q_1}{\partial z} = \left(\frac{k}{\alpha_1}\right)^{2/3} \alpha_1 . \tag{2.155}$$

When the transmitter is in the second layer and the receiver is in the first layer, the modal height-gain function is given by

$$\begin{aligned} E(n,z_R,z_T) = \pm R_2 W \left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} \Bigg\{ & \left[\mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33}) k_2(q_{23}) \right. \\ & \left. + h_2(q_{33}) k_2'(q_{23}) \right] \hat{k}_2(q_{11}) k_1(q_{2T}) k_1(q_{1R}) \\ & - \left[\mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33}) k_1(q_{23}) + h_2(q_{33}) k_1'(q_{23}) \right] \hat{k}_2(q_{11}) k_2(q_{2T}) k_1(q_{1R}) \\ & - \left[\mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33}) k_2(q_{23}) + h_2(q_{33}) k_2'(q_{23}) \right] \hat{k}_1(q_{11}) k_1(q_{2T}) k_2(q_{1R}) \\ & + \left[\mp \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33}) k_1(q_{23}) + h_2(q_{33}) k_1'(q_{23}) \right] \hat{k}_1(q_{11}) k_2(q_{2T}) k_2(q_{1R}) \Bigg\} \end{aligned}$$

$$(0 \le z_R \le z_2 \; ; \; z_2 \le z_T \le z_3) . \tag{2.156}$$

When the transmitter is in the second layer and the receiver is in the second layer, the modal height-gain function is

$$
\begin{aligned}
E(n,z_R,z_T) = \\
R_2 \Bigg\{ \Bigg[\Bigg\{ - \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33})k_2(q_{23}) + h_2(q_{33})k_2'(q_{23}) \Bigg\} \Bigg\{ \Bigg[-\left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k_2'(q_{22})k_2(q_{12}) \\
+ k_2(q_{22})k_2'(q_{12}) \Bigg] \hat{k}_1(q_{11}) \\
+ \left[\left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k_2'(q_{22})k_1(q_{12}) - k_2(q_{22})k_1'(q_{12}) \right] \hat{k}_2(q_{11}) \Bigg\} \Bigg] k_1(q_{2T})k_1(q_{2R}) \\
+ \Bigg[\Bigg\{ -\left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33})k_2(q_{23}) + h_2(q_{33})k_2'(q_{23}) \Bigg\} \Bigg\{ \Bigg[\left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k_1'(q_{22})k_2(q_{12}) \\
- k_1(q_{22})k_2'(q_{12}) \Bigg] \hat{k}_1(q_{11}) \\
+ \left[-\left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k_1'(q_{22})k_1(q_{12}) + k_1(q_{22})k_1'(q_{12}) \right] \hat{k}_2(q_{11}) \Bigg\} \Bigg] \Big\{ k_2(q_{2T})k_1(q_{2R}) \\
+ k_1(q_{2T})k_2(q_{2R}) \Big\} \\
+ \Bigg[\Bigg\{ -\left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} h_2'(q_{33})k_1(q_{23}) + h_2(q_{33})k_1'(q_{23}) \Bigg\} \Bigg\{ \Bigg[-\left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k_1'(q_{22})k_2(q_{12}) \\
+ k_1(q_{22})k_2'(q_{12}) \Bigg] \hat{k}_1(q_{11}) \\
+ \left[\left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k_1'(q_{22})k_1(q_{12}) - k_1(q_{22})k_1'(q_{12}) \right] \hat{k}_2(q_{11}) \Bigg\} \Bigg] k_2(q_{2T})k_2(q_{2R}) \Bigg\}
\end{aligned}
$$

$$(z_2 \le z_R \le z_3 \; ; \; z_2 \le z_T \le z_3 \; ; \; \alpha_2 > 0) \qquad (2.157)$$

or

$$E(n,z_R,z_T) =$$

$$R_2 \left\{ \left[\left\{ \left(\frac{\alpha_3}{|\alpha_2|} \right)^{1/3} h_2'(q_{33})k_2(q_{23}) + h_2(q_{33})k_2'(q_{23}) \right\} \left\{ \left[\left(\frac{|\alpha_2|}{\alpha_1} \right)^{1/3} k_2'(q_{22})k_2(q_{12}) \right.\right.\right.$$

$$\left. + k_2(q_{22})k_2'(q_{12}) \right] \hat{k}_1(q_{11}) + \left[- \left(\frac{|\alpha_2|}{\alpha_1} \right)^{1/3} k_2'(q_{22})k_1(q_{12}) \right.$$

$$\left.\left. - k_2(q_{22})k_1'(q_{12}) \right] \hat{k}_2(q_{11}) \right\} \left] k_1(q_{2T})k_1(q_{2R}) \right.$$

$$+ \left[\left\{ \left(\frac{\alpha_3}{|\alpha_2|} \right)^{1/3} h_2'(q_{33})k_1(q_{23}) + h_2(q_{33})k_1'(q_{23}) \right\} \left\{ \left[- \left(\frac{|\alpha_2|}{\alpha_1} \right)^{1/3} k_2'(q_{22})k_2(q_{12}) \right.\right.\right.$$

$$\left. - k_2(q_{22})k_2'(q_{12}) \right] \hat{k}_1(q_{11}) + \left[\left(\frac{|\alpha_2|}{\alpha_1} \right)^{1/3} k_2'(q_{22})k_1(q_{12}) \right.$$

$$\left.\left. + k_2(q_{22})k_1'(q_{12}) \right] \hat{k}_2(q_{11}) \right\} \left] \left\{ k_2(q_{2T})k_1(q_{2R}) + k_1(q_{2T})k_2(q_{2R}) \right\} \right.$$

$$+ \left[\left\{ \left(\frac{\alpha_3}{|\alpha_2|} \right)^{1/3} h_2'(q_{33})k_1(q_{23}) + h_2(q_{33})k_1'(q_{23}) \right\} \left\{ \left[\left(\frac{|\alpha_2|}{\alpha_1} \right)^{1/3} k_1'(q_{22})k_2(q_{12}) \right.\right.\right.$$

$$\left. + k_1(q_{22})k_2'(q_{12}) \right] \hat{k}_1(q_{11}) + \left[- \left(\frac{|\alpha_2|}{\alpha_1} \right)^{1/3} k_1'(q_{22})k_1(q_{12}) \right.$$

$$\left.\left. - k_1(q_{22})k_1'(q_{12}) \right] \hat{k}_2(q_{11}) \right\} \left] k_2(q_{2T})k_2(q_{2R}) \right\}$$

$$(z_2 \le z_R \le z_3 \; ; \; z_2 \le z_T \le z_3 \; ; \; \alpha_2 < 0) \; . \qquad (2.158)$$

When the transmitter is in the second layer and the receiver is in the third layer, the modal height-gain function is

$$E(n,z_R,z_T) = R_2 W \left\{ \left[\left\{ \mp \left(\frac{|\alpha_2|}{\alpha_1} \right)^{1/3} k_2'(q_{22})k_2(q_{12}) + k_2(q_{22})k_2'(q_{12}) \right\} \hat{k}_1(q_{11}) \right.\right.$$

$$\left. - \left\{ \mp \left(\frac{|\alpha_2|}{\alpha_1} \right)^{1/3} k_2'(q_{22})k_1(q_{12}) + k_2(q_{22})k_1'(q_{12}) \right\} \hat{k}_2(q_{11}) \right] k_1(q_{2T})h_2(q_{3R})$$

$$+ \left[- \left\{ \mp \left(\frac{|\alpha_2|}{\alpha_1} \right)^{1/3} k_1'(q_{22})k_2(q_{12}) + k_1(q_{22})k_2'(q_{12}) \right\} \hat{k}_1(q_{11}) \right.$$

$$+ \left\{ \mp \left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k_1'(q_{22})k_1(q_{12}) + k_1(q_{22})k_1'(q_{12}) \right\} \hat{k}_2(q_{11}) \Bigg] k_2(q_{2T})h_2(q_{3R}) \Bigg\}$$

$$(z_3 \leq z_R \; ; \; z_2 \leq z_T \leq z_3) \; . \tag{2.159}$$

In the above equations

$$R_2 = \frac{p}{Wq_2'} \; , \tag{2.160}$$

where

$$q_2' = \frac{\partial q_2}{\partial z} = \left(\frac{k}{|\alpha_2|}\right)^{2/3} \alpha_2 \; . \tag{2.161}$$

When the transmitter is in the third layer and the receiver is in the first layer, the modal height-gain function is given by

$$E(n,z_R,z_T) = - R_1 W^2 \{\hat{k}_1(q_{11})k_2(q_{1R}) - \hat{k}_2(q_{11})k_1(q_{1R})\}h_2(q_{3T})$$

$$(0 \leq z_R \leq z_2 \; ; \; z_3 \leq z_T) \; . \tag{2.162}$$

When the transmitter is in the third layer and the receiver is in the second layer, the modal height-gain function is

$$E(n,z_R,z_T) = \pm R_3 W \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3} \left\{ \left[\left\{ \mp \left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k'_2(q_{22})k_2(q_{12}) \right.\right.\right.$$

$$\left. + k_2(q_{22})k_2'(q_{12}) \right\} \hat{k}_1(q_{11})$$

$$\left. - \left\{ \mp \left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k'_2(q_{22})k_1(q_{12}) + k_2(q_{22})k_1'(q_{12}) \right\} \hat{k}_2(q_{11}) \right] h_2(q_{3T})k_1(q_{2R})$$

$$+ \left[- \left\{ \mp \left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3} k'_1(q_{22})k_2(q_{12}) + k_1(q_{22})k_2'(q_{12}) \right\} \hat{k}_1(q_{11}) \right.$$

$$+\left\{\mp\left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3}k_1'(q_{22})k_1(q_{12}) + k_1(q_{22})k_1'(q_{12})\right\}\hat{k}_2(q_{11})\Bigg]h_2(q_{3T})k_2(q_{2R})\Bigg\}$$

$$(z_2 \le z_R \le z_3 \; ; \; z_3 \le z_T) \; . \qquad (2.163)$$

When the transmitter is in the third layer and the receiver is in the third layer, the modal height-gain function is given by

$$E(n,z_R,z_T) = R_3\Bigg\{\left[\mp\left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3}k_1'(q_{22})k_2(q_{12}) + k_1(q_{22})k_2'(q_{12})\right]\left[-k_2'(q_{23})\left\{h_1(q_{33})\right.\right.$$

$$\left.-e^{-4\pi i/3}h_2(q_{33})\right\} \pm \left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3}k_2(q_{23})\left\{h_1'(q_{33})-e^{-4\pi i/3}h_2'(q_{33})\right\}\Bigg]\hat{k}_1(q_{11})$$

$$+\left[\mp\left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3}k_2'(q_{22})k_2(q_{12}) + k_2(q_{22})k_2'(q_{12})\right]\left[k_1'(q_{23})\left\{h_1(q_{33})-e^{-4\pi i/3}h_2(q_{33})\right\}\right.$$

$$\left.\mp\left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3}k_1(q_{23})\left\{h_1'(q_{33})-e^{-4\pi i/3}h_2'(q_{33})\right\}\right]\hat{k}_1(q_{11})$$

$$+\left[\mp\left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3}k_1'(q_{22})k_1(q_{12}) + k_1(q_{22})k_1'(q_{12})\right]\left[k_2'(q_{23})\left\{h_1(q_{33})-e^{-4\pi i/3}h_2(q_{33})\right\}\right.$$

$$\left.\mp\left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3}k_2(q_{23})\left\{h_1'(q_{33})-e^{-4\pi i/3}h_2'(q_{33})\right\}\right]\hat{k}_2(q_{11})$$

$$+\left[\mp\left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3}k_2'(q_{22})k_1(q_{12}) + k_2(q_{22})k_1'(q_{12})\right]\left[-k_1'(q_{23})\left\{h_1(q_{33})-e^{-4\pi i/3}h_2(q_{33})\right\}\right.$$

$$\left.\pm\left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3}k_1(q_{23})\left\{h_1'(q_{33})-e^{-4\pi i/3}h_2'(q_{33})\right\}\right]\hat{k}_2(q_{11})\Bigg\}h_2(q_{3T})h_2(q_{3R})$$

$$(z_3 \le z_R \; ; \; z_3 \le z_T) \; . \qquad (2.164)$$

In the above equations

$$R_3 = \frac{p}{Wq_3'} , \tag{2.165}$$

where

$$q_3' = \frac{\partial q_3}{\partial z} = \left(\frac{k}{\alpha_3}\right)^{2/3} \alpha_3 . \tag{2.166}$$

In all of the above equations for the modal height-gain function, the upper sign is to be taken when $\alpha_2 > 0$ and the lower sign when $\alpha_2 < 0$.

3. SURFACE ROUGHNESS

The preceding results have assumed that the earth's surface is perfectly smooth. Because surface roughness could have a significant effect upon the signal levels at large ranges, especially for frequencies above several gigahertz, the modifications necessary to incorporate surface roughness into the previous results will now be made.

One way to introduce surface roughness, or any loss mechanism that can be modeled as a lossy boundary, is the following [16]. Instead of the trilinear layer shown in figure 1, consider the layer shown in figure 3, where the infinitesimally thin bottom layer, $z_1 = 0 \leq z \leq z_p$, has a constant modified refractivity. In the limit $z_p \to z_1 = 0$ this profile is exactly the same as the one considered before. In the bottom layer $0 \leq z \leq z_p$, the function $\tilde{\Pi}(\rho,z)$ can be written as

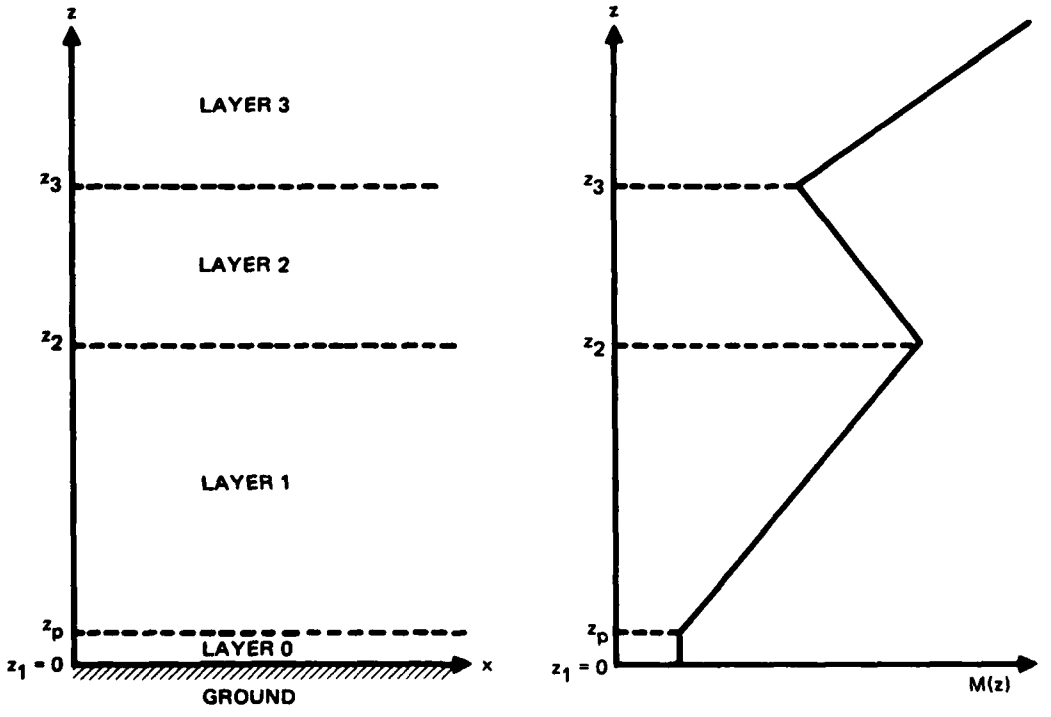

Figure 3. Modified refractivity profile used to introduce surface roughness loss into calculation of field strength. In layer 0, $\frac{dM}{dz} = 0$. The height z_p is infinitesimally small, and eventually $z_p \to 0$.

$$\tilde{\Pi}_o(\rho,z) = A_o(\rho)\{e^{i\mu z} + Re^{-i\mu z}\} , \tag{3.1}$$

where

$$\mu = k\sqrt{n_1^2(0) - \beta^2} \tag{3.2}$$

and

$$\beta^2 = \frac{\rho^2}{k^2} . \tag{3.3}$$

For a time dependence of $e^{+i\omega t}$ and with

$$Re\{\sqrt{n_1^2(0) - \beta^2}\} = Re\left\{\left(\frac{\alpha_1}{k}\right)^{1/3}\sqrt{q_{11}}\right\} > 0 , \tag{3.4}$$

the first term in (3.1) represents a down-going plane wave and the second term represents an up-going plane wave. The effect of surface roughness is introduced through the plane wave reflection coefficient R in equation (3.1). With the introduction of (3.1), the function $\tilde{\Pi}(\rho,z)$ is no longer required to satisfy the boundary condition that $\tilde{\Pi}(\rho,z)$ represent an outgoing decaying wave as $z \to -\infty$. In the other layers the solution $\tilde{\Pi}(\rho,z)$ is exactly as before:

$$\tilde{\Pi}_j(\rho,z) = \begin{cases} A_j(\rho)k_1(q_j) + B_j(\rho)k_2(q_j) + \tilde{\Pi}_j^{(P)}(\rho,q_j)\delta_{jM} & (j = 1,2) \\ \\ B_3(\rho)h_2(q_3) + \tilde{\Pi}_3^{(P)}(\rho,q_3)\delta_{3M} & (j = 3) , \end{cases} \tag{3.5}$$

where the particular solutions $\tilde{\Pi}_j^{(P)}$ are given by equations (2.40, 2.41).

The continuity conditions that $\tilde{\Pi}(\rho,z)$ now has to satisfy are

$$\tilde{\Pi}_o(\rho,z_p) = \tilde{\Pi}_1(\rho,z_p) , \tag{3.6}$$

$$\frac{d}{dz}\{\tilde{\Pi}_o(\rho,z_p)\} = \frac{d}{dz}\{\tilde{\Pi}_1(\rho,z_p)\} , \tag{3.7}$$

$$\tilde{\Pi}_j(\rho,z_{j+1}) = \tilde{\Pi}_{j+1}(\rho,z_{j+1}) \qquad (j = 1,2) , \tag{3.8}$$

and

$$\frac{d}{dz}\{\tilde{\Pi}_j(\rho,z_{j+1})\} = \frac{d}{dz}\{\tilde{\Pi}_{j+1}(\rho,z_{j+1})\} \qquad (j = 1,2)\ , \tag{3.9}$$

for both vertical and horizontal polarization. Substituting equations (3.1) and (3.5) into equations (3.6, 3.7) yields

$$A_o\left\{e^{i\mu z_p} + Re^{-i\mu z_p}\right\} = A_1 k_1(q_{1p}) + B_1 k_2(q_{1p}) + \tilde{\Pi}_1^{(P)}(\rho,q_{1p})\delta_{1M} \tag{3.10}$$

and

$$i\mu A_o\left\{e^{i\mu z_p} - Re^{-i\mu z_p}\right\} = \alpha_1 \left(\frac{k}{|\alpha_1|}\right)^{2/3} A_1 k_1'(q_{1p}) + \alpha_1 \left(\frac{k}{|\alpha_1|}\right)^{2/3} B_1 k_2'(q_{1p})$$
$$+ \frac{d}{dz}\{\tilde{\Pi}_1^{(P)}(\rho,q_{1p})\}\delta_{1M}\ , \tag{3.11}$$

where

$$q_{1p} = q_1(z_p) = \left(\frac{k}{|\alpha_1|}\right)^{2/3}[1 + i\eta + \alpha_1(z_p - H) - \beta^2]\ . \tag{3.12}$$

In the limit $z_p \to 0$, equations (3.10, 3.11) become

$$A_o(1 + R) = A_1 k_1(q_{11}) + B_1 k_2(q_{11}) + \tilde{\Pi}_1^{(P)}(\rho,q_{11})\delta_{1M} \tag{3.13}$$

and

$$i\mu A_o(1 - R) = \alpha_1\left(\frac{k}{|\alpha_1|}\right)^{2/3} A_1 k_1'(q_{11}) + \alpha_1 \left(\frac{k}{|\alpha_1|}\right)^{2/3} B_1 k_2'(q_{11})$$
$$+ \frac{d}{dz}\left\{\tilde{\Pi}_1^{(P)}(\rho,q_{11})\right\}\delta_{1M}\ . \tag{3.14}$$

Eliminating A_o from equations (3.13, 3.14) finally yields

$$A_1\left\{k_1'(q_{11}) - \frac{i\mu}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\left(\frac{1-R}{1+R}\right) k_1(q_{11})\right\} + B_1\left\{k_2'(q_{11}) - \frac{i\mu}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\left(\frac{1-R}{1+R}\right)k_2(q_{11})\right\}$$

$$= -\frac{1}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\frac{d}{dz}\left\{\tilde{\Pi}_1^{(P)}(\rho,q_{11})\right\}\delta_{1M} + \frac{i\mu}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\left(\frac{1-R}{1+R}\right)\tilde{\Pi}_1^{(P)}(\rho,q_{11})\delta_{1M} \ . \tag{3.15}$$

The continuity conditions (3.8, 3.9) yield equations (2.53-2.56) as before. In comparison of equations (2.52) and (3.15) it can be seen that the net effect of the modification to include surface roughness is the replacement

$$\gamma \to \mu\,\frac{1-R}{1+R} \ . \tag{3.16}$$

Therefore, all of the previous results hold if the functions $\hat{k}_1(q_{11})$ and $\hat{k}_2(q_{11})$ are defined to be

$$\hat{k}_1(q_{11}) = k_1'(q_{11}) - G_R k_1(q_{11}) \tag{3.17}$$

and

$$\hat{k}_2(q_{11}) = k_2'(q_{11}) - G_R k_1(q_{11}) \ , \tag{3.18}$$

where

$$G_R = \frac{i\mu}{\alpha_1}\left(\frac{|\alpha_1|}{k}\right)^{2/3}\left(\frac{1-R}{1+R}\right)$$

$$= i\left(\frac{k}{\alpha_1}\right)^{1/3}\sqrt{n_1^2(0) - \beta^2}\left(\frac{1-R}{1+R}\right) \qquad (\alpha_1 > 0) \ . \tag{3.19}$$

When reflection occurs from a surface with a Gaussian distribution of bump heights, the reflection coefficient R can be represented as the product of the smooth surface plane wave reflection coefficient and a surface roughness factor [17]. For horizontal polarization, the reflection coefficient R is given by

$$R = \begin{cases} R_H \exp\left[-2k^2\delta^2\left(\frac{\alpha_1}{k}\right)^{2/3} q_{11}\right] & \mathrm{Re}(q_{11}) \geq 0 \\ R_H & \mathrm{Re}(q_{11}) < 0 \ , \end{cases} \tag{3.20}$$

where δ is the root mean square surface bump height and R_H is the Fresnel plane wave reflection coefficient for horizontal polarization

$$R_H = \frac{[n_1^2(0) - \beta^2]^{1/2} - [n_g^2 - \beta^2]^{1/2}}{[n_1^2(0) - \beta^2]^{1/2} + [n_g^2 - \beta^2]^{1/2}} \tag{3.21}$$

When $\mathrm{Re}(q_{11}) < 0$, the magnitude of the exponential factor $\exp\left[-2k^2\delta^2\left(\frac{\alpha_1}{k}\right)^{2/3} q_{11}\right]$ is greater than one. Since modes with $\mathrm{Re}(q_{11}) < 0$ correspond to well-trapped modes and should be unaffected by the surface, the exponential factor is not included in R when $\mathrm{Re}(q_{11}) < 0$.

In the case of vertical polarization, R is given by

$$R = \begin{cases} R_V \exp\left[-2k^2\delta^2\left(\frac{\alpha_1}{k}\right)^{2/3} q_{11}\right] & \mathrm{Re}(q_{11}) \geq 0 \\ R_V & \mathrm{Re}(q_{11}) < 0 \ , \end{cases} \tag{3.22}$$

where R_V is the Fresnel plane wave reflection coefficient for vertical polarization

$$R_V = \frac{n_g^2[n_1^2(0) - \beta^2]^{1/2} - n_1^2(0)[n_g^2 - \beta^2]^{1/2}}{n_g^2[n_1^2(0) - \beta^2]^{1/2} + n_1^2(0)[n_g^2 - \beta^2]^{1/2}} \ . \tag{3.23}$$

The limits of G_R as the rms bump height δ approaches zero are

$$G_{RH} = \frac{i\mu}{\alpha_1}\left(\frac{\alpha_1}{k}\right)^{2/3}\left(\frac{1-R_H}{1+R_H}\right) = i[n_1^2(0) - \beta^2]^{1/2}\left(\frac{k}{\alpha_1}\right)^{1/3}\left(\frac{1-R_H}{1+R_H}\right)$$

$$\rightarrow \frac{i\gamma}{\alpha_1}\left(\frac{\alpha_1}{k}\right)^{2/3} = i[n_g^2 - \beta^2]^{1/2}\left(\frac{k}{\alpha_1}\right)^{1/3} = G_{SH} \tag{3.24}$$

for horizontal polarization and

$$G_{RV} = \frac{i\mu}{\alpha_1}\left(\frac{\alpha_1}{k}\right)^{2/3}\left(\frac{1-R_V}{1+R_V}\right) = i[n_1^2(0) - \beta^2]^{1/2}\left(\frac{k}{\alpha_1}\right)^{1/3}\left(\frac{1-R_V}{1+R_V}\right)$$

$$\rightarrow \frac{i\gamma}{\alpha_1}\left(\frac{\alpha_1}{k}\right)^{2/3}\frac{n_1^2(0)}{n_g^2} = i[n_g^2 - \beta^2]^{1/2}\left(\frac{k}{\alpha_1}\right)^{1/3}\frac{n_1^2(0)}{n_g^2} = G_{SV} \ . \qquad (3.25)$$

Thus, the above results for reflection from a rough surface contain the results for reflection from a smooth surface as a special case.

The coefficient G_R and its derivative $\frac{\partial G_R}{\partial q_{11}}$ are needed when evaluating the modal function and its derivative. For horizontal polarization, G_{RH} can be written as

$$G_{RH} = i\left(\frac{k}{\alpha_1}\right)^{1/3}\frac{a(q_{11})}{b(q_{11})} \ , \qquad (3.26)$$

where

$$a(q_{11}) = \left(\frac{\alpha_1}{k}\right)^{1/3}(q_{11})^{1/2}\tanh\left(\frac{\phi}{2}\right) + \left[\tau + \left(\frac{\alpha_1}{k}\right)^{2/3}q_{11}\right]^{1/2} \qquad (3.27)$$

and

$$b(q_{11}) = 1 + \left(\frac{k}{\alpha_1}\right)^{1/3}(q_{11})^{-1/2}\tanh\left(\frac{\phi}{2}\right)\left[\tau + \left(\frac{\alpha_1}{k}\right)^{2/3}q_{11}\right]^{1/2} \qquad (3.28)$$

with

$$\tau = n_g^2 - 1 - i\eta - \alpha_1 H \qquad (3.29)$$

and

$$\phi = 2k^2\delta^2\left(\frac{\alpha_1}{k}\right)^{2/3}q_{11} \ . \qquad (3.30)$$

For small values of $|q_{11}|$, $a(q_{11})$ and $b(q_{11})$ have the series expansions

$$a(q_{11}) = \left(\frac{\alpha_1}{k}\right)^{1/3}(q_{11})^{1/2}\left\{k^2\delta^2\left(\frac{\alpha_1}{k}\right)^{2/3}q_{11}\right.$$

$$\left. - \frac{1}{3}k^6\delta^6\left(\frac{\alpha_1}{k}\right)^2 q_{11}^3 + \frac{2}{15}k^{10}\delta^{10}\left(\frac{\alpha_1}{k}\right)^{10/3}q_{11}^5 - \ \ldots\right\}$$

$$+ \tau^{1/2}\left\{1 + \frac{1}{2\tau}\left(\frac{\alpha_1}{k}\right)^{2/3} q_{11} - \frac{1}{8\tau^2}\left(\frac{\alpha_1}{k}\right)^{4/3} q^2{}_{11} + \frac{1}{16\tau^3}\left(\frac{\alpha_1}{k}\right)^2 q_{11}^3\right.$$

$$\left. - \frac{5}{128\tau^4}\left(\frac{\alpha_1}{k}\right)^{8/3} q_{11}^4 + \frac{7}{256\tau^5}\left(\frac{\alpha_1}{k}\right)^{10/3} q_{11}^5 - \ldots \right\} \qquad (3.31)$$

and

$$b(q_{11}) = 1 + \tau^{1/2}(q_{11})^{1/2}\left\{k^2\delta^2\left(\frac{\alpha_1}{k}\right)^{1/3} + \frac{1}{2\tau}k^2\delta^2\left(\frac{\alpha_1}{k}\right)q_{11} - \left[\frac{1}{3}k^6\delta^6 + \right.\right.$$

$$\left. + \frac{1}{8\tau^2}k^2\delta^2\right]\left(\frac{\alpha_1}{k}\right)^{5/3} q_{11}^2 + \left[-\frac{1}{6\tau}k^6\delta^6 + \frac{1}{16\tau^3}k^2\delta^2\right]\left(\frac{\alpha_1}{k}\right)^{7/3} q_{11}^3$$

$$+ \left[\frac{2}{15}k^{10}\delta^{10} + \frac{1}{24\tau^2}k^6\delta^6 - \frac{5}{128\tau^4}k^2\delta^2\right]\left(\frac{\alpha_1}{k}\right)^3 q_{11}^4 + \left[\frac{1}{15\tau}k^{10}\delta^{10}\right.$$

$$\left.\left. - \frac{1}{48\tau^3}k^6\delta^6 + \frac{7}{256\tau^5}k^2\delta^2\right]\left(\frac{\alpha_1}{k}\right)^{11/3} q_{11}^5 + \ldots\right\}. \qquad (3.32)$$

The derivative of G_{RH} with respect to q_{11} is given by

$$\frac{\partial G_{RH}}{\partial q_{11}} = i\left(\frac{k}{\alpha_1}\right)^{1/3}\left\{\frac{1}{b}\ \frac{\partial a}{\partial q_{11}} - \ \frac{a}{b^2}\ \frac{\partial b}{\partial q_{11}}\right\}, \qquad (3.33)$$

where

$$\frac{\partial a}{\partial q_{11}} = \frac{1}{2}(q_{11})^{-1/2}\left(\frac{\alpha_1}{k}\right)^{1/3} \tanh\left(\frac{\phi}{2}\right) + (q_{11})^{1/2}\left(\frac{\alpha_1}{k}\right)k^2\delta^2\ \mathrm{sech}^2\left(\frac{\phi}{2}\right)$$

$$+ \frac{1}{2}\left(\frac{\alpha_1}{k}\right)^{2/3}\left[\tau + \left(\frac{\alpha_1}{k}\right)^{2/3} q_{11}\right]^{-1/2} \qquad (3.34)$$

and

$$\frac{\partial b}{\partial q_{11}} = -\frac{1}{2}\left(\frac{k}{\alpha_1}\right)^{1/3} q_{11}^{-3/2} \tanh\left(\frac{\phi}{2}\right)\left[\tau + \left(\frac{\alpha_1}{k}\right)^{2/3} q_{11}\right]^{1/2}$$

$$+ \frac{1}{2}q_{11}^{-1/2}\left(\frac{\alpha_1}{k}\right)^{1/3} \tanh\left(\frac{\phi}{2}\right)\left[\tau + \left(\frac{\alpha_1}{k}\right)^{2/3} q_{11}\right]^{-1/2}$$

$$+ q_{11}^{-1/2} k^2 \delta^2 \left(\frac{\alpha_1}{k}\right)^{1/3} \operatorname{sech}^2\left(\frac{\phi}{2}\right) \left[\tau + \left(\frac{\alpha_1}{k}\right)^{2/3} q_{11}\right]^{1/2} . \qquad (3.35)$$

For small values of $|q_{11}|$, $\frac{\partial a}{\partial q_{11}}$ and $\frac{\partial b}{\partial q_{11}}$ have the series expansions

$$\frac{\partial a}{\partial q_{11}} = \left(\frac{\alpha_1}{k}\right)^{1/3} q_{11}^{1/2} \left\{\frac{3}{2} k^2 \delta^2 \left(\frac{\alpha_1}{k}\right)^{2/3} - \frac{7}{6} k^6 \delta^6 \left(\frac{\alpha_1}{k}\right)^2 q_{11}^2 \right.$$

$$\left. + \frac{11}{15} k^{10} \delta^{10} \left(\frac{\alpha_1}{k}\right)^{10/3} q_{11}^4 - \ldots \right\} + \tau^{1/2} \left\{\frac{1}{2\tau} \left(\frac{\alpha_1}{k}\right)^{2/3} \right.$$

$$- \frac{1}{4\tau^2} \left(\frac{\alpha_1}{k}\right)^{4/3} q_{11} + \frac{3}{16\tau^3} \left(\frac{\alpha_1}{k}\right)^2 q^2{}_{11} - \frac{5}{32\tau^4} \left(\frac{\alpha_1}{k}\right)^{8/3} q_{11}^3$$

$$\left. + \frac{35}{256\tau^5} \left(\frac{\alpha_1}{k}\right)^{10/3} q_{11}^4 - \ldots \right\} \qquad (3.36)$$

and

$$\frac{\partial b}{\partial q_{11}} = \tau^{1/2} q_{11}^{-1/2} \left\{\frac{1}{2} k^2 \delta^2 \left(\frac{\alpha_1}{k}\right)^{1/3} + \frac{3}{4\tau} k^2 \delta^2 \left(\frac{\alpha_1}{k}\right) q_{11} \right.$$

$$- \frac{5}{2} \left[\frac{1}{3} k^6 \delta^6 + \frac{1}{8\tau^2} k^2 \delta^2\right] \left(\frac{\alpha_1}{k}\right)^{5/3} q^2{}_{11}$$

$$+ \frac{7}{2} \left[-\frac{1}{6\tau} k^6 \delta^6 + \frac{1}{16\tau^3} k^2 \delta^2\right] \left(\frac{\alpha_1}{k}\right)^{7/3} q_{11}^3 + \frac{9}{2} \left[\frac{2}{15} k^{10} \delta^{10} + \frac{1}{24\tau^2} k^6 \delta^6 \right.$$

$$\left. - \frac{5}{128\tau^4} k^2 \delta^2\right] \left(\frac{\alpha_1}{k}\right)^3 q_{11}^4 + \frac{11}{2} \left[\frac{1}{15\tau} k^{10} \delta^{10} \right.$$

$$\left. - \frac{1}{48\tau^3} k^6 \delta^6 + \frac{7}{256\tau^5} k^2 \delta^2\right] \left(\frac{\alpha_1}{k}\right)^{11/3} q_{11}^5 + \ldots \right\} . \qquad (3.37)$$

The corresponding results for G_{RV} are

$$G_{RV} = i\left(\frac{k}{\alpha_1}\right)^{1/3} \frac{\tilde{a}(q_{11})}{\tilde{b}(q_{11})} \ , \tag{3.38}$$

where

$$\tilde{a}(q_{11}) = \left(\frac{\alpha_1}{k}\right)^{1/3} q_{11}^{1/2} \tanh\left(\frac{\phi}{2}\right) + \frac{n_1^2(0)}{n_g^2}\left[\tau + \left(\frac{\alpha_1}{k}\right)^{2/3} q_{11}\right]^{1/2} \tag{3.39}$$

and

$$\tilde{b}(q_{11}) = 1 + \frac{n_1^2(0)}{n_g^2} q_{11}^{-1/2}\left(\frac{k}{\alpha_1}\right)^{1/3} \tanh\left(\frac{\phi}{2}\right)\left[\tau + \left(\frac{\alpha_1}{k}\right)^{2/3} q_{11}\right]^{1/2} . \tag{3.40}$$

For small values of $|q_{11}|$, $\tilde{a}(q_{11})$ and $\tilde{b}(q_{11})$ have the expansions

$$\begin{aligned}\tilde{a}(q_{11}) = {} & \left(\frac{\alpha_1}{k}\right)^{1/3} q_{11}^{1/2}\left\{k^2\delta^2\left(\frac{\alpha_1}{k}\right)^{2/3} q_{11} - \frac{1}{3}k^6\delta^6\left(\frac{\alpha_1}{k}\right)^2 q_{11}^3\right. \\ & \left. + \frac{2}{15}k^{10}\delta^{10}\left(\frac{\alpha_1}{k}\right)^{10/3} q_{11}^5 - \cdots\right\} \\ & + \frac{n_1^2(0)}{n_g^2}\tau^{1/2}\left\{1 + \frac{1}{2\tau}\left(\frac{\alpha_1}{k}\right)^{2/3} q_{11} - \frac{1}{8\tau^2}\left(\frac{\alpha_1}{k}\right)^{4/3} q_{11}^2 + \frac{1}{16\tau^3}\left(\frac{\alpha_1}{k}\right)^2 q_{11}^3\right. \\ & \left. - \frac{5}{128\tau^4}\left(\frac{\alpha_1}{k}\right)^{8/3} q_{11}^4 + \frac{7}{256\tau^5}\left(\frac{\alpha_1}{k}\right)^{10/3} q_{11}^5 - \cdots\right\}\end{aligned} \tag{3.41}$$

and

$$\begin{aligned}\tilde{b}(q_{11}) = {} & 1 + \frac{n_1^2(0)}{n_g^2}\tau^{1/2} q_{11}^{1/2}\left\{k^2\delta^2\left(\frac{\alpha_1}{k}\right)^{1/3} + \frac{1}{2\tau}k^2\delta^2\left(\frac{\alpha_1}{k}\right) q_{11} - \left[\frac{1}{3}k^6\delta^6\right.\right. \\ & \left.\left. + \frac{1}{8\tau^2}k^2\delta^2\right]\left(\frac{\alpha_1}{k}\right)^{5/3} q_{11}^2 + \left[-\frac{1}{6\tau}k^6\delta^6 + \frac{1}{16\tau^3}k^2\delta^2\right]\left(\frac{\alpha_1}{k}\right)^{7/3} q_{11}^3\right. \\ & \left. + \left[\frac{2}{15}k^{10}\delta^{10} + \frac{1}{24\tau^2}k^6\delta^6 - \frac{5}{128\tau^4}k^2\delta^2\right]\left(\frac{\alpha_1}{k}\right)^3 q_{11}^4 + \left[\frac{1}{15\tau}k^{10}\delta^{10}\right.\right.\end{aligned}$$

$$- \frac{1}{48\tau^3}k^6\delta^6 + \frac{7}{256\tau^5}k^2\delta^2\Big]\left(\frac{\alpha_1}{k}\right)^{11/3} q_{11}^5 + \ldots\Big\} . \qquad (3.42)$$

The derivative of G_{RV} with respect to q_{11} is equal to

$$\frac{\partial G_{RV}}{\partial q_{11}} = i\left(\frac{k}{\alpha_1}\right)^{1/3}\left\{\frac{1}{\tilde{b}}\frac{\partial \tilde{a}}{\partial q_{11}} - \frac{\tilde{a}}{\tilde{b}^2}\frac{\partial \tilde{b}}{\partial q_{11}}\right\} , \qquad (3.43)$$

where

$$\frac{\partial \tilde{a}}{\partial q_{11}} = \frac{1}{2}q_{11}^{-1/2}\left(\frac{\alpha_1}{k}\right)^{1/3}\tanh\left(\frac{\phi}{2}\right) + k^2\delta^2\frac{\alpha_1}{k}q_{11}^{1/2}\operatorname{sech}^2\left(\frac{\phi}{2}\right)$$

$$+ \frac{1}{2}\frac{n_f^2(0)}{n_g^2}\left(\frac{\alpha_1}{k}\right)^{2/3}\left[\tau + \left(\frac{\alpha_1}{k}\right)^{2/3} q_{11}\right]^{-1/2} \qquad (3.44)$$

and

$$\frac{\partial \tilde{b}}{\partial q_{11}} = \frac{n_f^2(0)}{n_g^2}\left\{-\frac{1}{2}q_{11}^{-3/2}\left(\frac{k}{\alpha_1}\right)^{1/3}\tanh\left(\frac{\phi}{2}\right)\left[\tau + \left(\frac{\alpha_1}{k}\right)^{2/3} q_{11}\right]^{1/2}\right.$$

$$+ \frac{1}{2}q_{11}^{-1/2}\left(\frac{\alpha_1}{k}\right)^{1/3}\tanh\left(\frac{\phi}{2}\right)\left[\tau + \left(\frac{\alpha_1}{k}\right)^{2/3} q_{11}\right]^{-1/2}$$

$$\left. + k^2\delta^2 q_{11}^{-1/2}\left(\frac{\alpha_1}{k}\right)^{1/3}\operatorname{sech}^2\left(\frac{\phi}{2}\right)\left[\tau + \left(\frac{\alpha_1}{k}\right)^{2/3} q_{11}\right]^{1/2}\right\}. \qquad (3.45)$$

For small values of $|q_{11}|$, $\frac{\partial \tilde{a}}{\partial q_{11}}$ and $\frac{\partial \tilde{b}}{\partial q_{11}}$ have the series expansions

$$\frac{\partial \tilde{a}}{\partial q_{11}} = \left(\frac{\alpha_1}{k}\right)^{1/3} q_{11}^{1/2}\left\{\frac{3}{2}k^2\delta^2\left(\frac{\alpha_1}{k}\right)^{2/3} - \frac{7}{6}k^6\delta^6\left(\frac{\alpha_1}{k}\right)^2 q_{11}^2\right.$$

$$\left. + \frac{11}{15}k^{10}\delta^{10}\left(\frac{\alpha_1}{k}\right)^{10/3} q_{11}^4 - \ldots\right\} + \frac{n_f^2(0)}{n_g^2}\tau^{1/2}\left\{\frac{1}{2\tau}\left(\frac{\alpha_1}{k}\right)^{2/3}\right.$$

$$- \frac{1}{4\tau^2}\left(\frac{\alpha_1}{k}\right)^{4/3} q_{11} + \frac{3}{16\tau^3}\left(\frac{\alpha_1}{k}\right)^{2} q_{11}^2 - \frac{5}{32\tau^4}\left(\frac{\alpha_1}{k}\right)^{8/3} q_{11}^3$$

$$+ \frac{35}{256\tau^5}\left(\frac{\alpha_1}{k}\right)^{10/3} q_{11}^4 - \ldots \Bigg\} \tag{3.46}$$

and

$$\frac{\partial \tilde{b}}{\partial q_{11}} = \frac{n_f^2(0)}{n_g^2}\, \tau^{1/2} q_{11}^{-1/2} \Bigg\{ \frac{1}{2}k^2\delta^2\left(\frac{\alpha_1}{k}\right)^{1/3} + \frac{3}{4\tau}k^2\delta^2\left(\frac{\alpha_1}{k}\right) q_{11}$$

$$- \frac{5}{2}\left[\frac{1}{3}k^6\delta^6 + \frac{1}{8\tau^2}k^2\delta^2\right]\left(\frac{\alpha_1}{k}\right)^{5/3} q_{11}^2 + \frac{7}{2}\left[-\frac{1}{6\tau}k^6\delta^6\right.$$

$$\left. + \frac{1}{16\tau^3}k^2\delta^2\right]\left(\frac{\alpha_1}{k}\right)^{7/3} q_{11}^3 + \frac{9}{2}\left[\frac{2}{15}k^{10}\delta^{10} + \frac{1}{24\tau^2}k^6\delta^6\right.$$

$$\left. - \frac{5}{128\tau^4}k^2\delta^2\right]\left(\frac{\alpha_1}{k}\right)^{3} q_{11}^4 + \frac{11}{2}\left[\frac{1}{15\tau}k^{10}\delta^{10}\right.$$

$$\left. - \frac{1}{48\tau^3}k^6\delta^6 + \frac{7}{256\tau^5}k^2\delta^2\right]\left(\frac{\alpha_1}{k}\right)^{11/3} q_{11}^5 + \ldots \Bigg\} . \tag{3.47}$$

4. COMPLEX INDEX OF REFRACTION OF SEAWATER

In the waveguide formulation for tropospheric ducting of electromagnetic waves, one of the physical parameters needed for evaluating the modal function and height-gain functions is the complex index of refraction of seawater. The complex index of refraction of seawater is a function of frequency, temperature, and salinity. The temperature of the ocean's surface lies between -2°C and 32°C, with about half of the ocean's surface being warmer than 20°C. Although the salinity of the ocean near the mouth of large rivers may approach zero, the salinity of most water in the open ocean lies between 33 and 37 grams of salt per kilogram of seawater, and 35 g salt/kg seawater is often chosen as standard [18].

The complex index of refraction of seawater, n_g, can be related to an effective relative dielectric function, ε_{eff}, and an effective conductivity, σ_{eff}, through

$$n_g^2 = \varepsilon_{eff} - i \frac{\sigma_{eff}}{\omega \varepsilon_o} , \tag{4.1}$$

where ε_o is the permittivity of free space and has the value

$$\varepsilon_o = 8.85434 \times 10^{-12} \text{ F/m} , \tag{4.2}$$

while $f = \frac{\omega}{2\pi}$ is the frequency at which n_g is to be evaluated. The effective dielectric function and the effective conductivity are related to the complex dielectric function

$$\varepsilon = \varepsilon' - i\varepsilon'' \tag{4.3}$$

and the ionic conductivity, σ, through

$$\varepsilon_{eff} = \varepsilon' \tag{4.4}$$

and

$$\sigma_{eff} = \sigma + \omega \varepsilon_o \varepsilon'' . \tag{4.5}$$

The above equations assume the ionic conductivity to have an imaginary part of zero. For frequencies much less than 10^4 GHz this is a good approximation.

The theory of an ideal polar dielectric in an alternating electromagnetic field was given by Debye [19] in terms of a single relaxation time. While Debye's derivation assumes a spherical molecule whose free rotations are resisted by a viscous force, all that is needed is the assumption of exponential relaxation of the polarization [20]. In the Debye theory the complex dielectric function is given by

$$\varepsilon = \varepsilon' - i\varepsilon'' , \tag{4.6}$$

where

$$\varepsilon' = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + (\lambda_s/\lambda)^2}$$

$$= \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + \omega^2\tau^2} \tag{4.7}$$

and

$$\varepsilon'' = \frac{(\varepsilon_s - \varepsilon_\infty)(\lambda_s/\lambda)}{1 + (\lambda_s/\lambda)^2} \tag{4.8}$$

$$= \frac{(\varepsilon_s - \varepsilon_\infty)\omega\tau}{1 + \omega^2\tau^2} \quad . \tag{4.9}$$

In equations (4.7, 4.8), ε_∞ is that part of the dielectric function which is due to the atomic and electronic polarization and is assumed to be real and independent of frequency, and ε_s is the static ($\omega \to 0$) value of the dielectric function. The wavelength of the alternating electromagnetic field is λ, while λ_s, the characteristic wavelength, is the wavelength corresponding to the frequency $\frac{1}{\tau}$, where τ is the dielectric relaxation time. Equations (4.6-4.8) represent the fall in the value of the dielectric function from $\varepsilon_s(\omega \to 0)$ to $\varepsilon_\infty(\omega \to \infty)$, the fall being accompanied by a single broad absorption band in the neighborhood of the characteristic wavelength, λ_s. The representation of the variation with frequency of the dielectric function in terms of a single dielectric relaxation time τ should be valid for frequencies up to 300 GHz [21]. For higher frequencies more than one relaxation time will be needed.

Some of the earliest experimental work on the dielectric properties of pure water and aqueous ionic solutions was performed by JB Hasted,

CH Collie, and DM Ritson [22, 23]. For pure water they measured the value of the complex dielectric function at the wavelengths, λ, of 10.00, 3.213, and 1.274 cm (f = 2.998, 9.331, and 23.53 GHz) for temperatures between 0°C and 75°C. For temperatures in the range 0°C to 40°C, the least-square fit to their values for the characteristic wavelength and static dielectric constant is

$$\varepsilon_{\infty}^{(P)} = 5.5 \ , \tag{4.9}$$

$$\lambda_s^{(P)}(T) = 3.338 - 0.11417T + 0.0022T^2 - 1.833 \times 10^{-6}T^3 \ , \tag{4.10}$$

and

$$\varepsilon_s^{(P)}(T) = 88.192 - 0.405T + 0.0007T^2 \ , \tag{4.11}$$

where T is in degrees Celsius and λ_s is in centimeters. Equations (4.9-4.11) give the parameters in the Debye equation for the complex dielectric function of pure water as a function of temperature. For seawater there is a lowering of the value of the dielectric function from its pure water value due to saturation of the dielectric in the neighborhood of the salt ions. Hasted, Ritson, and Collie [23] also measured the complex dielectric function of ionic solutions. For NaCl solutions less than 1.5 molar, their experimental values could be fit by the Debye formula with λ_s and ε_s, as functions of temperature and salinity, given by

$$\lambda_s(T,S) = \lambda_s^{(P)}(T) - 0.08\ c \tag{4.12}$$

and

$$\varepsilon_s(T,S) = \varepsilon_s^{(P)}(T) - 11.0\ c \ , \tag{4.13}$$

where λ_s is again in centimeters and c is the salt concentration in moles per liter. They assumed that the value of ε_{∞} for NaCl solutions was 5.5, the same as for pure water.

A more recent fit of the Debye model parameters has been obtained by Klein and Swift [24]. Assuming that

$$\varepsilon_{\infty} = 4.9 \tag{4.14}$$

for both pure water and seawater, they found that the static dielectric constant of pure water is given by

$$\varepsilon_s^{(P)}(T) = 88.045 - 0.4147T + 6.295 \times 10^{-4}T^2 + 1.075 \times 10^{-5}T^3 \ , \quad (4.15)$$

while the static dielectric constant of seawater is given by

$$\varepsilon_s(T,S) = \varepsilon_s(T)a(S,T) \ , \quad (4.16)$$

where

$$\varepsilon_s(T) = 87.134 - 1.949 \times 10^{-1}T - 1.276 \times 10^{-2}T^2 + 2.491 \times 10^{-4}T^3 \quad (4.17)$$

and

$$a(S,T) = 1.000 + 1.613 \times 10^{-5}ST - 3.656 \times 10^{-3}S + 3.210 \times 10^{-5}S^2 - 4.232 \times 10^{-7}S^3 \ . \quad (4.18)$$

The temperature T is in degrees Celsius and the salinity S is in grams of salt per kilogram of seawater. When S = 0 the expressions (4.16-4.18) for the static dielectric constant of seawater do not reduce to the expression (4.15) for the static dielectric constant of pure water. This may be due to the presence of sulphates and nitrates in the seawater samples measured [24]. The dielectric relaxation time of seawater in the Klein and Swift model is given by

$$\tau(T,S) = \tau(T,0)b(S,T) \ , \quad (4.19)$$

where $\tau(T,0)$ is the pure water value of the relaxation time and is equal to

$$\tau(T,0) = 1.768 \times 10^{-11} - 6.086 \times 10^{-13}T + 1.104 \times 10^{-14}T^2 - 8.111 \times 10^{-17}T^3 \ , \quad (4.20)$$

and

$$b(S,T) = 1.000 + 2.282 \times 10^{-5}ST - 7.638 \times 10^{-4}S - 7.760 \times 10^{-6}S^2 + 1.105 \times 10^{-8}S^3 \ . \quad (4.21)$$

The dielectric relaxation time is in seconds. The ionic conductivity of seawater is given by

$$\sigma(T,S) = \sigma(25,S)\exp(-\Delta\beta) \ , \quad (4.22)$$

where

$$\Delta = 25 - T \ , \quad (4.23)$$

$$\beta = 2.033 \times 10^{-2} + 1.266 \times 10^{-4}\Delta + 2.464 \times 10^{-6}\Delta^2 - S[1.849 \times 10^{-5} - 2.551 \times 10^{-7}\Delta + 2.551 \times 10^{-8}\Delta^2] \ , \quad (4.24)$$

and

$$\sigma(25,S) = S[0.182521 - 1.46192 \times 10^{-3}S + 2.09324 \times 10^{-5}S^2 - 1.28205 \times 10^{-7}S^3] \ . \qquad (4.25)$$

The conductivity σ is in siemens per meter.

5. SHELLMAN-MORFITT ROOT-FINDING ROUTINE

When the mode sum formulation of the tropospheric ducting problem is used, the most difficult part of the field strength calculations is determining the zeros of the modal function. For the trilinear model of the tropospheric index of refraction considered in this document, the modal function is given by equation (2.124):

$$||\underset{\sim}{\Delta}|| = \mp\left(\frac{\alpha_3}{|\alpha_2|}\right)^{1/3}\left\{\mp\left(\frac{|\alpha_2|}{\alpha_1}\right)^{1/3}\zeta(q_{11})\chi(q_{11}) + \psi(q_{11})\phi(q_{11})\right\} \quad , \qquad (5.1)$$

where $\zeta(q_{11})$, $\chi(q_{11})$, $\psi(q_{11})$, and $\phi(q_{11})$ are given by equations (2.125-2.128) and involve sums and products of Airy functions. It is not possible to solve equation (5.1) analytically for the complex zeros, q_{11}. Therefore, numerical methods must be used. Several algorithms exist for finding the complex roots of transcendental equations [4, 25-27]. In XWVG, the complex root-finding routine of Shellman and Morfitt is used to find the zeros of the modal function (5.1). This routine was also used by Goodhart and Pappert [28] and Marcus [1, 2] for finding the zeros of modal functions for linear segmented models of tropospheric refractivity.

The root-finding routine of Shellman and Morfitt locates all the simple zeros of an analytic function in a prescribed rectangular region of the complex plane. Initial guesses for the location of the zeros within the prescribed rectangular region of the complex plane are not required. The root-finding method of Shellman and Morfitt is based on the theorem [29]:

> Theorem: Let f(z) be a meromorphic function, i.e., analytic everywhere in the complex z-plane except at isolated poles. Let Γ be a closed contour in the z-plane such that there are no zeros or poles of f(z) on Γ. Then, the accumulated phase change, $\Delta\phi$, of f(z) around Γ traversed in a counterclockwise direction is equal to
>
> $$(\Delta\phi)_\Gamma = 2\pi(N_z - N_p) \; . \qquad (5.2)$$
>
> N_z is the number of zeros enclosed by Γ and N_p is the number of poles enclosed by Γ. When determining N_z and N_p, n-th order zeros and poles are counted n times.

For a function that has only zeros in the region enclosed by Γ so that $N_p = 0$, this theorem implies that every phase contour associated with each zero crosses Γ.

Consider the analytic function f(z). In terms of its modulus and phase, f(z) can be written as

$$f(z) = \{[\mathrm{Re}(f)]^2 + [\mathrm{Im}(f)]^2\}^{\frac{1}{2}} e^{i\theta} , \tag{5.3}$$

where θ is the phase of f and is given by

$$\theta = \tan^{-1}\left\{\frac{\mathrm{Im}(f)}{\mathrm{Re}(f)}\right\} . \tag{5.4}$$

If f(z) has no poles or branch points, the curve for the constant phase value $\theta = \theta_c$ radiating from a zero of f(z) must cross a closed contour Γ enclosing that zero at least once. Furthermore, no other zero of f(z) may be on this phase curve. See figure 4. A curve of constant phase, e.g., $\theta_c = 0$, which crosses the closed contour Γ may be followed until it leads to a zero of f(z) or until the phase curve again crosses the contour Γ. A zero of f(z) may be determined by the crossing of the curves

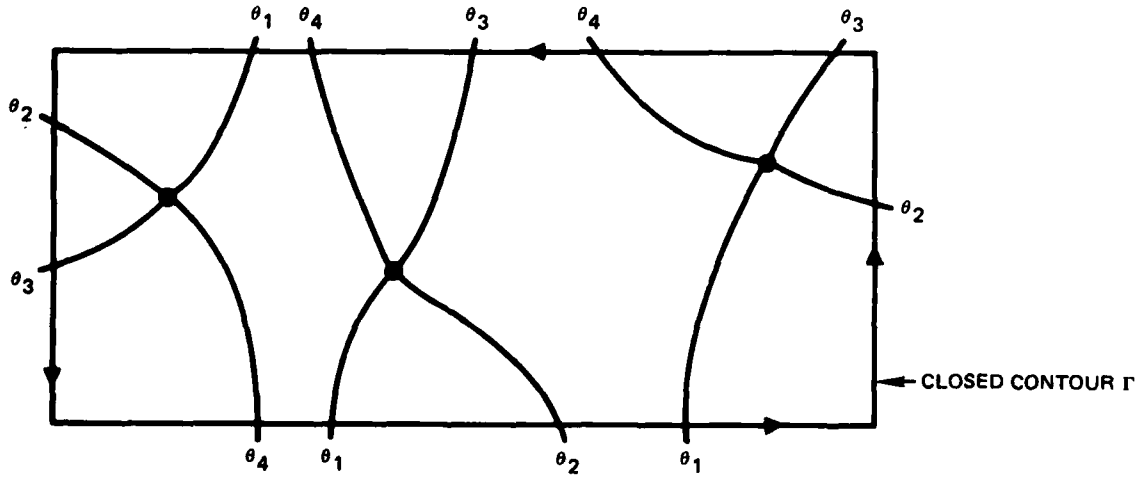

Figure 4. Constant phase curves for an analytic function with three simple zeros in the region enclosed by the rectangular contour Γ.

$$\mathrm{Im}(f) = 0 \qquad (\theta_c = 0 \text{ or } \pi) \tag{5.5}$$

and

$$\mathrm{Re}(f) = 0 \qquad \left(\theta_c = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}\right) . \tag{5.6}$$

The routine of Shellman and Morfitt will search a specified rectangular region of the complex z-plane for the simple zeros of the analytic function f(z). Let the edges of the specified search rectangle be denoted by t_L, t_R, t_T, and t_B, where t_L and t_R denote the value of the real part of z at the left and right edges of the search rectangle and t_T and t_B denote the value of the imaginary part of z at the top and bottom edges of the search rectangle. This search rectangle is divided into small mesh squares in which each mesh square side has length δ. Restrictions upon the size of δ will be discussed later. A new search rectangle is generated from the specified search rectangle. The left edge of this new search rectangle is given by

$$J_L = \mathrm{Int}\left(\frac{t_L}{\delta}\right) , \tag{5.7}$$

where Int(x) denotes the integer part of x. The other edges, J_R, J_T, and J_B, of this new search rectangle are similarly obtained. Thus, the new search rectangle is given in terms of mesh units, where one mesh unit is of length δ. Since truncation may occur in evaluating J_L, J_R, J_T, and J_B, this new search rectangle is made one mesh size larger on all sides in order to ensure that the original specified search rectangle falls completely within the new search rectangle. If the original specified search rectangle is near a branch cut or other discontinuity of f(z), this expansion of the search region could cause problems.

Consider an individual mesh square in the new expanded search rectangle as shown in figure 5. A local mesh coordinate system, where each unit is of length δ, may be set up as shown in figure 6. The value of f(z) evaluated in the local mesh coordinates of mesh square k is denoted by $f^{(k)}(\chi_k,\omega_k)$. Thus, as shown in figure 6, the value of f(z) at the origin of mesh square k is denoted by $f^{(k)}(0,0)$. Note that for each mesh square the lower left-hand corner is taken as the origin in that mesh square's local coordinate system. The values of f(z) at the corners of mesh square 1 are

$$f^{(1)}(0,1) = f(J_L\delta + iJ_T\delta) , \tag{5.8}$$

$$f^{(1)}(0,0) = f(J_L\delta + i[J_T - 1]\delta) , \tag{5.9}$$

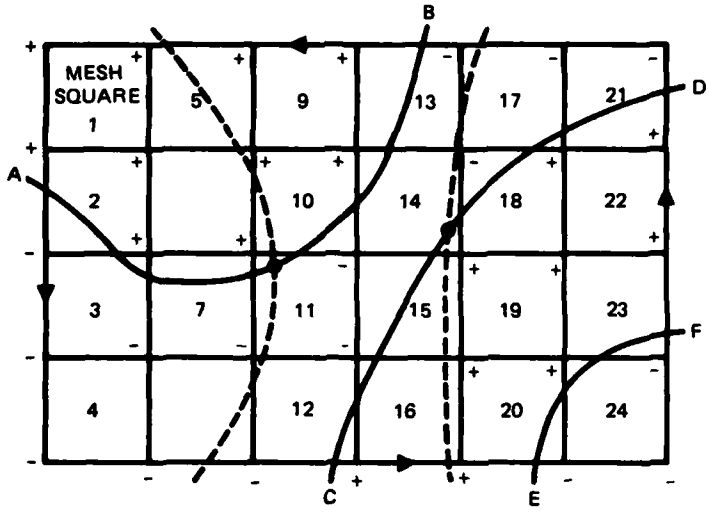

Figure 5. Illustration of search rectangle and mesh grid set up in the complex z-plane by the Shellman-Morfitt root-finding routine. The plus and minus signs at the corners of the mesh squares indicate the sign of Im(f). Along the solid curves Im(f) = 0; while along the dashed curves, Re(f) = 0. The zeros of f(z) are located at the intersection of the Im(f) = 0 and Re(f) = 0 curves.

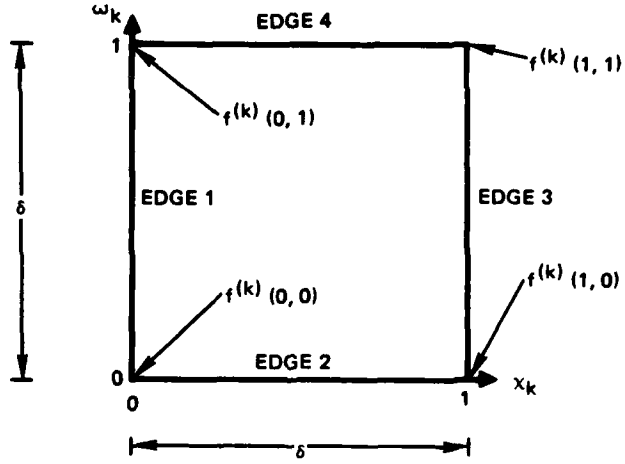

Figure 6. Local mesh coordinate system and numbering of the edges of mesh square k. The value of f(z) evaluated at the corners of mesh square k are denoted by $f^{(k)}(x_k, \omega_k)$, where x_k and ω_k equal 0 or 1 at the corners.

$$f^{(1)}(1,0) = f([J_L + 1]\delta + i[J_T - 1]\delta) \ , \tag{5.10}$$

and

$$f^{(1)}(1,1) = f([J_L + 1]\delta + iJ_T\delta) \ . \tag{5.11}$$

The values of f(z) at the corners of all other mesh squares are obtained similarly.

A basic assumption concerning f(z) made in the algorithm of Shellman and Morfitt is that along the edges of every mesh square k, Re(f) and Im(f) are linear functions of the local mesh coordinates χ_k and ω_k. This implies that in mesh square k, f(z) can be written as

$$f^{(k)}(\chi_k,\omega_k) = a_k + b_k\omega_k + c_k\chi_k + d_k\omega_k\chi_k \ , \tag{5.12}$$

where a_k, b_k, c_k, and d_k are complex constants. This assumption puts an upper limit on the size of δ that can be used. If δ is too large, the assumption that f(z) has a linear variation along the edges of the mesh squares breaks down. Thus, δ should be chosen small enough so that equation (5.12) is a good approximation to f(z). In order to reduce computer run times, however, δ should not be excessively small. Note that equation (5.12) implies that in the mesh square, the curves Im(f) = 0 and Re(f) = 0 are hyperbolic curves with vertical and horizontal asymptotes.

In order to further describe the workings of the root-finding routine of Shellman and Morfitt, use will be made of the example shown in figure 5. The zeros of f(z) in the new expanded search rectangle are looked for by examining the imaginary part of f(z) counterclockwise around the rectangular contour. Intermediate steps of investigating mesh squares are also carried out counterclockwise. For the example shown in figure 5, the edges of the search rectangle are examined for crossings of phase curves along which Im(f) = 0, starting with mesh square 1 edge 1. The values $\mathrm{Im}[f^{(1)}(0,1)]$ and $\mathrm{Im}[f^{(1)}(0,0)]$ are computed. If, as shown in figure 5, the signs of these two values are the same, then there can be no Im(f) = 0 curve passing through edge 1 of mesh square 1. This is a consequence of assuming that Im(f) varies linearly along the edges of the mesh squares.

Next, mesh square 2 is investigated. To obtain the values $\mathrm{Im}[f^{(2)}(0,1)]$ and $\mathrm{Im}[f^{(2)}(0,0)]$, use is made of the relationship

$$\mathrm{Im}[f^{(2)}(0,1)] = \mathrm{Im}[f^{(1)}(0,0)]. \tag{5.13}$$

The value of $\mathrm{Im}[f^{(2)}(0,0)]$ will be a new computed value. If the signs of $\mathrm{Im}[f^{(2)}(0,1)]$ and $\mathrm{Im}[f^{(2)}(0,0)]$ are opposite, as shown in figure 5, then an $\mathrm{Im}(f) = 0$ curve enters mesh square 2 somewhere along edge 1. Because of the assumed linearity of $\mathrm{Im}(f)$ along the mesh edge, the $\mathrm{Im}(f) = 0$ curve AB crosses edge 1 at

$$x_2 = 0 \tag{5.14}$$

and

$$\omega_2 = - \frac{\mathrm{Im}[f^{(2)}(0,0)]}{\mathrm{Im}[f^{(2)}(0,1)] - \mathrm{Im}[f^{(2)}(0,0)]} \quad . \tag{5.15}$$

The problem is now to find whether a zero of $f(z)$ lies within this mesh square and where the curve AB exits from the mesh square. First, the values $\mathrm{Im}[f^{(2)}(1,0)]$ and $\mathrm{Im}[f^{(2)}(1,1)]$ are obtained. The values of $\mathrm{Im}(f)$ at all four corners of the mesh square are now known. Next, a test is made to determine whether there are one or two $\mathrm{Im}(f) = 0$ curves entering and leaving the mesh square. If the values $\mathrm{Im}[f^{(k)}(0,0)]$ and $\mathrm{Im}[f^{(k)}(1,1)]$ have the same sign, e.g., minus, and the values $\mathrm{Im}[f^{(k)}(0,1)]$ and $\mathrm{Im}[f^{(k)}(1,0)]$ both have the opposite sign, e.g., plus, then there are two $\mathrm{Im}(f) = 0$ curves entering and leaving the mesh square. This situation will arise in mesh square 14 of figure 5. Otherwise, there is only one $\mathrm{Im}(f) = 0$ curve entering and leaving the mesh square, as in mesh square 2. Next, a test is performed to determine whether there is at least one $\mathrm{Re}(f) = 0$ curve entering and leaving the mesh square. If the values $\mathrm{Re}[f^{(k)}(0,0)]$, $\mathrm{Re}[f^{(k)}(1,0)]$, $\mathrm{Re}[f^{(k)}(1,1)]$, and $\mathrm{Re}[f^{(k)}(0,1)]$ all have the same sign, then there are no $\mathrm{Re}(f) = 0$ curves passing through the mesh square. If these values of $\mathrm{Re}(f)$ do not all have the

same sign, then there is at least one Re(f) = 0 curve entering and leaving the mesh square. Mesh square 2 does not have a Re(f) = 0 curve passing through it and therefore does not contain a zero of f(z). The edge through which the Im(f) = 0 curve exits the mesh square can be determined by using the signs of Im(f) at the corners of the mesh square. In figure 5, the Im(f) = 0 curve AB exits mesh square 2 across edge 2, which means it enters mesh square 3 across edge 4. The local mesh coordinates of mesh square 3 where the curve AB enters mesh square 3 can be determined by using the values

$$\mathrm{Im}[f^{(3)}(0,1)] = \mathrm{Im}[f^{(2)}(0,0)] \tag{5.16}$$

and

$$\mathrm{Im}[f^{(3)}(1,1)] = \mathrm{Im}[f^{(2)}(1,0)] \tag{5.17}$$

together with the assumed linearity of Im(f) along the mesh square edge.

The Im(f) = 0 curve is traced through the search rectangle, and each mesh square through which it passes is checked for zeros of f(z). In figure 5, the Im(f) = 0 curve AB is followed through mesh squares 2, 3, 7, 11, 10, 14, and 13. The tests for a zero of f(z) are made in mesh squares 3 and 7, with the same results as in mesh square 2. In mesh square 11, however, Im(f) = 0 and Re(f) = 0 curves both pass through the mesh square. On the basis of the values of f(z) at the corners of the mesh square and the assumed linearity of f(z) along the mesh square edges, the value of f(z) at any point within the mesh square can be determined from

$$f^{(k)}(\chi_k,\omega_k) = a_k + b_k\omega_k + c_k\chi_k + d_k\omega_k\chi_k \ , \tag{5.18}$$

where

$$a_k = f^{(k)}(0,0) \ , \tag{5.19}$$

$$b_k = f^{(k)}(0,1) - f^{(k)}(0,0) \ , \tag{5.20}$$

$$c_k = f^{(k)}(1,0) - f^{(k)}(0,0) \ , \tag{5.21}$$

and

$$d_k = f^{(k)}(0,0) + f^{(k)}(1,1) - f^{(k)}(0,1) - f^{(k)}(1,0) \ . \tag{5.22}$$

The point of intersection, if any, of the curves Im(f) = 0 and Re(f) = 0 can then be determined. The mathematical details are presented in the report by Morfitt and Shellman [4].

After determining and storing the approximate location of the zero of f(z) in mesh square 11, the contour AB is followed out of mesh square 11 into mesh square 10, then into mesh square 14. Two Im(f) = 0 curves and one Re(f) = 0 curve pass through mesh square 14. A solution for a zero of f(z) in a mesh square is accepted if it lies on the Im(f) = 0 curve currently being followed [4]. When the contour AB is followed, the zero in mesh square 14 will be ignored since it falls on the Im(f) = 0 curve CD. When curve CD is traced through the search rectangle, this zero will be found again and stored. From mesh square 14, the contour AB is traced to mesh square 13, where it exits the search rectangle. Information that gives the mesh square and edge where the contour AB exits the search rectangle is stored. Identifying these exit points makes it possible to avoid later following the same Im(f) = 0 curve through the search rectangle. For example, if the Im(f) = 0 trace was later allowed to reenter the search rectangle at B, the same zero of f(z) in mesh square 11 would be found.

After the Im(f) = 0 curve trace exits at point B, the search for zeros of f(z) is continued by looking for more crossings of Im(f) = 0 curves with the rectangular contour. Since mesh square 2 was the last mesh square examined for such crossings, the next mesh square to be examined will be mesh square 3. The next crossing of an Im(f) = 0 curve with the rectangular contour will occur in mesh square 12. Since this crossing point is different from the previously stored exit point B, the Im(f) = 0 curve CD will be followed through the search rectangle. The zero of f(z) in mesh square 14 will again be located; and since it lies on the Im(f) = 0 curve currently being traced, its location is stored. The curve CD is followed the rest of the way through the search rectangle, and the exit point D is stored. The search along the rectangular contour then continues at mesh square 16. At mesh square 20, a new crossing of an Im(f) = 0 curve with the rectangular contour is detected. The curve EF is followed through the search rectangle, and the exit point F is stored. No zeros of f(z) are found along this curve. The rectangular contour search continues at mesh 24 edge 2 and goes until mesh square 23, where another

crossing of an Im(f) = 0 curve with the rectangular contour is found. But since this crossing point is the exit point found when following the curve EF, it is ignored and the rectangular contour search continues. The rectangular contour search will continue as described above until the search finally gets to mesh square 1 edge 4, where it stops.

The zeros of f(z) found by the above search must be considered as approximate, since the approximation

$$f^{(k)}(\chi_k,\omega_k) = a_k + b_k\omega_k + c_k\chi_k + d_k\omega_k\chi_k \tag{5.23}$$

was used in each mesh square containing a zero, to solve for the location of the zero. These approximate zero locations can be used, however, as starting values in a Newton-Raphson iteration

$$z_{i+1} = z_i - \frac{f(z_i)}{f'(z_i)}, \tag{5.24}$$

in order to locate the zeros to within a preassigned tolerance ε.

When the Shellman-Morfitt root-finding routine was used to find zeros of the modal function (5.1), problems were encountered in locating the zeros that lie along the real q_{11}-axis. These zeros correspond to well-trapped modes. The cause of this difficulty is the fact that, as originally implemented, a mesh line would always fall on the real q_{11}-axis. The Shellman-Morfitt algorithm assumes that the zeros do not lie on a mesh line. Since some of the zeros of the modal function fall on or very close to the real q_{11}-axis, this assumption was not always true and difficulties ensued. To overcome this, a small offset was used to move the mesh lines off the real axis. No further difficulties were encountered once this was done.

6. SOLUTIONS OF STOKES' DIFFERENTIAL EQUATION

When the modified refractivity profile is approximated by linear segments, in the waveguide formulation of propagation of electromagnetic waves in tropospheric ducts, the modal equation and height-gain functions are expressible in terms of the solutions of Stokes' differential equation

$$\frac{d^2w}{d\xi^2} + \xi w(\xi) = 0 \ . \tag{6.1}$$

Since this is a second-order linear differential equation, there exist two linearly independent solutions $w_1(\xi)$ and $w_2(\xi)$. Any other solution to (6.1) can be expressed as a linear combination of $w_1(\xi)$ and $w_2(\xi)$:

$$y(\xi) = c_1 w_1(\xi) + c_2 w_2(\xi) \ . \tag{6.2}$$

One commonly chosen set of two linearly independent solutions of (6.1) consists of the Airy functions [30]

$$w_1(\xi) = Ai(-\xi) \tag{6.3}$$

and

$$w_2(\xi) = Bi(-\xi) \ . \tag{6.4}$$

The Wronskian for this set of linearly independent solutions is

$$W\{Ai(-\xi), Bi(-\xi)\} = Ai(-\xi) \frac{d}{d\xi} [Bi(-\xi)] - \frac{d}{d\xi}[Ai(-\xi)]Bi(-\xi)$$

$$= - \frac{1}{\pi} \ . \tag{6.5}$$

While $Ai(-\xi)$ and $Bi(-\xi)$ are analytically linearly independent everywhere in the complex ξ-plane, they are numerically linearly independent only along the real axis and in the sector $|ph\xi| < \frac{\pi}{3}$. The advantage of this choice of solutions of (6.1) is that when ξ is real, both $Ai(-\xi)$ and $Bi(-\xi)$ are real. As $-\xi \to +\infty$, $Ai(-\xi)$ decays exponentially while $Bi(-\xi)$ grows exponentially. As $-\xi \to -\infty$, both $Ai(-\xi)$ and $Bi(-\xi)$ are decaying oscillatory functions of the same amplitude and with interlacing zeros.

Another possible set of linearly independent solutions of (6.1) consists of [30]

$$w_1(\xi) = Ai(-\xi) \tag{6.6}$$

and

$$w_2(\xi) = Ai(-\xi e^{2\pi i/3}) \ . \tag{6.7}$$

The Wronskian corresponding to this choice of linearly independent solutions is

$$W\{Ai(-\xi),\ Ai(-\xi e^{2\pi i/3})\} = -\frac{1}{2\pi} e^{-\pi i/6} \ . \tag{6.8}$$

The functions $Ai(-\xi)$ and $Ai(-\xi e^{2\pi i/3})$ are numerically linearly independent in the upper half of the complex ξ-plane.

A third possible choice for the linearly independent solutions of (6.1) consists of

$$w_1(\xi) = Ai(-\xi) \tag{6.9}$$

and

$$w_1(\xi) = Ai(-\xi e^{-2\pi i/3}) \ . \tag{6.10}$$

The Wronskian of these functions is

$$W\{Ai(-\xi),\ Ai(-\xi e^{-2\pi i/3})\} = -\frac{1}{2\pi} e^{\pi i/6} \ . \tag{6.11}$$

The functions $Ai(-\xi)$ and $Ai(-\xi e^{-2\pi i/3})$ are numerically linearly independent in the lower half of the complex ξ-plane.

In radio propagation work other choices for the two linearly independent solutions of Stokes' differential equation are frequently used. One such set of functions is the modified Hankel functions of order one-third [13], $h_1(\xi)$ and $h_2(\xi)$. The modified Hankel functions and their derivatives are related to the Airy functions by

$$\begin{aligned} h_1(\xi) &= (12)^{1/6} e^{-\pi i/6} [Ai(-\xi) - iBi(-\xi)] \\ &= -2i(12)^{1/6} Ai(-\xi e^{2\pi i/3}) \ , \end{aligned} \tag{6.12}$$

$$h_2(\xi) = (12)^{1/6} e^{\pi i/6} [Ai(-\xi) + iBi(-\xi)]$$

$$= 2i(12)^{1/6} Ai(-\xi e^{-2\pi i/3}) , \tag{6.13}$$

$$h_1'(\xi) = -(12)^{1/6} e^{-\pi i/6} [Ai'(-\xi) - iBi'(-\xi)]$$

$$= 2i(12)^{1/6} e^{2\pi i/3} Ai'(-\xi e^{2\pi i/3}) , \tag{6.14}$$

and

$$h_2'(\xi) = -(12)^{1/6} e^{\pi i/6} [Ai'(-\xi) + iBi'(-\xi)]$$

$$= -2i(12)^{1/6} e^{-2\pi i/3} Ai'(-\xi e^{-2\pi i/3}) , \tag{6.15}$$

where the prime denotes the derivative with respect to the argument. The Wronskian of $h_1(\xi)$ and $h_2(\xi)$ is

$$W\{h_1(\xi), h_2(\xi)\} = -\frac{4i}{\pi} \left(\frac{3}{2}\right)^{1/3} . \tag{6.16}$$

Marcus [1, 2] uses the functions $k_1(\xi)$ and $k_2(\xi)$ as the two linearly independent solutions of (6.1). These functions and their derivatives are related to the modified Hankel functions and the Airy functions by

$$k_1(\xi) = h_1(\xi)$$

$$= -2i(12)^{1/6} Ai(-\xi e^{2\pi i/3}) , \tag{6.17}$$

$$k_2(\xi) = h_2(\xi) - e^{4\pi i/3} h_1(\xi)$$

$$= 2(12)^{1/6} e^{\pi i/6} Ai(-\xi) , \tag{6.18}$$

$$k_1'(\xi) = h_1'(\xi)$$

$$= 2i(12)^{1/6} e^{2\pi i/3} Ai'(-\xi e^{2\pi i/3}) , \tag{6.19}$$

and

$$k_2'(\xi) = h_2'(\xi) - e^{4\pi i/3} h_1'(\xi)$$

$$= -2(12)^{1/6} e^{\pi i/6} Ai'(-\xi) . \tag{6.20}$$

The Wronskian of $k_1(\xi)$ and $k_2(\xi)$ is

$$W\{k_1(\xi),\ k_2(\xi)\} = -\frac{4i}{\pi}\left(\frac{3}{2}\right)^{1/3} . \tag{6.21}$$

The choice of which set of linearly independent solutions of Stokes' differential equation to use is dictated by the boundary conditions and numerical considerations. The arguments of the Airy functions appearing in the solution (2.32) for $\tilde{\Pi}(\rho,q_j)$ are $q_j(z)$. Since $\mathrm{Im}(q_{11})$ is proportional to the attenuation rate of the electromagnetic signal, from physical considerations $\mathrm{Im}(q_{11}) > 0$. This, together with equations (2.30) and (2.129) implies that $\mathrm{Im}[q_j(z)] > 0$. Therefore, the Airy functions occurring in $\tilde{\Pi}(\rho,q_j)$ are all evalutated in the upper half of the complex q-plane. In the upper half of the complex plane a numerically satisfactory pair of linearly independent solutions of Stokes' differential equation consists of the Airy functions $\mathrm{Ai}(-\xi)$ and $\mathrm{Ai}(-\xi e^{2\pi i/3})$, or equivalently, $k_1(\xi)$ and $k_2(\xi)$.

The solution $\tilde{\Pi}(\rho,z)$ to the tropospheric ducting problem has to satisfy the boundary condition that as $z \to +\infty$, $\tilde{\Pi}(\rho,z)$ must represent an outgoing wave with time dependence $e^{i\omega t}$. A solution of Stokes' differential equation that asymptotically satisfies this requirement is $\mathrm{Ai}(-\xi e^{-2\pi i/3})$. Therefore, in order to satisfy the outgoing wave condition in the topmost layer of the modified refractivity profile, $\mathrm{Ai}(-\xi e^{-2\pi i/3})$ and $\mathrm{Ai}(-\xi)$, [or equivalently, $h_2(\xi)$ and $h_1(\xi)-e^{-4\pi i/3}h_2(\xi)$] are chosen as the linearly independent set of solutions of Stokes' differential equation. While this set of solutions does satisfy the outgoing wave boundary condition, it is not a numerically satisfactory set of solutions in the upper half of the complex q-plane. For this reason this set of solutions is used only in the topmost layer of the modified refractivity profile.

While asymptotic expansions of the Airy functions Ai(z) are available in the literature [30-32], none of these papers gives a rigorous upper bound for the magnitude of the error associated with these expansions. These bounds are useful for the writing of subroutines to evaluate the Airy functions. Error bounds for the asymptotic expansions of the Airy functions are next obtained by using the asymptotic expansions of the Hankel functions and modified Bessel functions and their error bounds.

The Airy function Ai(z) and its derivative are related to the Hankel functions by

$$\mathrm{Ai}(-z) = \frac{1}{2}\left(\frac{z}{3}\right)^{1/2}\left[e^{\pi i/6}H_{1/3}^{(1)}(\xi) + e^{-\pi i/6}H_{1/3}^{(2)}(\xi)\right] \tag{6.22}$$

and

$$\mathrm{Ai}'(-z) = \frac{1}{2}\left(\frac{z}{\sqrt{3}}\right)\left[e^{-\pi i/6}H_{2/3}^{(1)}(\xi) + e^{\pi i/6}H_{2/3}^{(2)}(\xi)\right], \tag{6.23}$$

where

$$\xi = \frac{2}{3}z^{3/2}. \tag{6.24}$$

The Hankel functions have the asymptotic expansions [33]

$$H_\nu^{(1)}(\xi) = \left(\frac{2}{\pi\xi}\right)^{1/2}\exp\left(i\xi - \frac{1}{2}\nu\pi i - \frac{1}{4}\pi i\right)\left\{\sum_{k=0}^{n-1} i^k A_k(\nu)\xi^{-k} + \eta_n^{(1)}(\xi,\nu)\right\} \qquad (-\pi + \delta \le \mathrm{ph}\xi \le 2\pi - \delta) \tag{6.25}$$

and

$$H_\nu^{(2)}(\xi) = \left(\frac{2}{\pi\xi}\right)^{1/2}\exp\left(-i\xi + \frac{1}{2}\nu\pi i + \frac{1}{4}\pi i\right)\left\{\sum_{k=0}^{n-1} (-i)^k A_k(\nu)\xi^{-k} + \eta_n^{(2)}(\xi,\nu)\right\} \qquad (-2\pi + \delta \le \mathrm{ph}\xi \le \pi - \delta), \tag{6.26}$$

where $\delta > 0$. The coefficients $A_k(\nu)$ are given by

$$A_0(v) = 1 \tag{6.27}$$

and

$$A_k(v) = \frac{[4v^2 - 1^2][4v^2 - 3^2] \cdots [4v^2 - (2k-1)^2]}{8^k\, k!} \quad (k \geq 1) \ . \tag{6.28}$$

The error term $\eta_n^{(1)}(\xi,v)$ is bounded by

$$|\eta_n^{(1)}(\xi,v)| \leq 2|A_n(v)|\phi_n^{(1)}(\xi)\exp\left\{|v^2 - \tfrac{1}{4}|\phi_1^{(1)}(\xi)\right\} , \tag{6.29}$$

where the error control function $\phi_n^{(1)}(\xi)$ has the bounds

$$\phi_n^{(1)}(\xi) \leq \begin{cases} |\xi|^{-n} & (0 \leq \mathrm{ph}\ \xi \leq \pi) \\ \chi(n)\ |\xi|^{-n} & \left(-\frac{\pi}{2} \leq \mathrm{ph}\xi \leq 0;\ \pi \leq \mathrm{ph}\xi \leq \frac{3\pi}{2}\right) \\ 2\chi(n)|\mathrm{Im}(\xi)|^{-n} & \left(-\pi + \delta \leq \mathrm{ph}\xi \leq -\frac{\pi}{2};\ \frac{3\pi}{2} \leq \mathrm{ph}\xi \leq 2\pi - \delta\right) \end{cases} \tag{6.30}$$

and

$$\chi(n) = \pi^{1/2}\ \frac{\Gamma\left(\frac{1}{2}n + 1\right)}{\Gamma\left(\frac{1}{2}n + \frac{1}{2}\right)} \ . \tag{6.31}$$

The error term $\eta_n^{(2)}(\xi,v)$ is bounded by

$$|\eta_n^{(2)}(\xi,v)| \leq 2|A_n(v)|\phi_n^{(2)}(\xi)\exp\left\{|v^2 - \tfrac{1}{4}|\phi_1^{(2)}(\xi)\right\} , \tag{6.32}$$

where the error control function $\phi_n^{(2)}(\xi)$ is bounded by

$$\phi_n^{(2)}(\xi) \leq \begin{cases} |\xi|^{-n} & (-\pi \leq \mathrm{ph}\xi \leq 0) \\ \chi(n)|\xi|^{-n} & (0 \leq \mathrm{ph}\xi \leq \frac{\pi}{2};\ -\frac{3\pi}{2} \leq \mathrm{ph}\xi \leq -\pi) \\ 2\chi(n)|\mathrm{Im}(\xi)|^{-n} & \left(\frac{\pi}{2} \leq \mathrm{ph}\xi \leq \pi - \delta\ ;\ -2\pi + \delta \leq \mathrm{ph}\xi \leq -\frac{3\pi}{2}\right) . \end{cases} \tag{6.33}$$

When $v = \frac{1}{3}$, equations (6.25, 6.26) become

$$H_{1/3}^{(1)}(\xi) = \left(\frac{2}{\pi\xi}\right)^{1/2} \exp\left(i\xi - \frac{5\pi i}{12}\right)\left\{\sum_{k=0}^{n-1} i^k A_k\left(\frac{1}{3}\right)\xi^{-k} + \eta_n^{(1)}(\xi, \tfrac{1}{3})\right\}$$

$$(-\pi + \delta \leq \mathrm{ph}\xi \leq 2\pi - \delta)$$

$$= \left(\frac{3}{\pi}\right)^{1/2} z^{-3/4} \exp\left(i\frac{2}{3}z^{3/2} - \frac{5\pi i}{12}\right)\left\{\sum_{k=0}^{n-1} (-i)^k C_k z^{-3k/2} + \eta_n^{(1)}\left(\frac{2}{3}z^{3/2}, \tfrac{1}{3}\right)\right\}$$

$$(-\frac{2\pi}{3} + \delta' \leq \mathrm{ph}z \leq \frac{4\pi}{3} - \delta') \qquad (6.34)$$

and

$$H_{1/3}^{(2)}(\xi) = \left(\frac{2}{\pi\xi}\right)^{1/2} \exp\left(-i\xi + \frac{5\pi i}{12}\right)\left\{\sum_{k=0}^{n-1} (-i)^k A_k\left(\frac{1}{3}\right)\xi^{-k} + \eta_n^{(2)}\left(\xi, \tfrac{1}{3}\right)\right\}$$

$$(-2\pi + \delta \leq \mathrm{ph}\xi \leq \pi - \delta)$$

$$= \left(\frac{3}{\pi}\right)^{1/2} z^{-3/4} \exp\left(-i\frac{2}{3}z^{3/2} + \frac{5\pi i}{12}\right)\left\{\sum_{k=0}^{n-1} i^k C_k z^{-3k/2} + \eta_n^{(2)}\left(\frac{2}{3}z^{3/2}, \tfrac{1}{3}\right)\right\}$$

$$(-\frac{4\pi}{3} + \delta' \leq \mathrm{ph}z \leq \frac{2\pi}{3} - \delta') \; . \qquad (6.35)$$

The coefficients $A_k\left(\frac{1}{3}\right)$ and C_k are given by

$$A_o\left(\tfrac{1}{3}\right) = C_o = 1 \qquad (6.36)$$

and

$$A_k\left(\tfrac{1}{3}\right) = (-1)^k \frac{2^k}{3^k} C_k$$

$$= (-1)^k \frac{[9 - 4][81 - 4]\cdots[9(2k - 1)^2 - 4]}{3^{2k}\, 2^{3k}\, k!} \qquad (k \geq 1) \; . \qquad (6.37)$$

The error terms $\eta_n^{(1)}\left(\xi, \frac{1}{3}\right)$ and $\eta_n^{(2)}\left(\xi, \frac{1}{3}\right)$ are bounded by

$$\left|\eta_n^{(1)}\left(\xi, \tfrac{1}{3}\right)\right| \leq \frac{2^{n+1}}{3^n} C_n \phi_n^{(1)}(\xi) \exp\left\{\frac{5}{36}\, \phi_1^{(1)}(\xi)\right\} \qquad (6.38)$$

and

$$|\eta_n^{(2)}(\xi,\tfrac{1}{3})| \le \frac{2^{n+1}}{3^n} C_n \phi_n^{(2)}(\xi) \exp\left\{\frac{5}{36}\phi_1^{(2)}(\xi)\right\}. \tag{6.39}$$

Using the above results in equation (6.22) yields the asymptotic expansion

$$\mathrm{Ai}(-z) = \tfrac{1}{2}\pi^{-1/2} z^{-1/4}\left\{\exp\left(i\xi - \tfrac{1}{4}\pi i\right)\left[\sum_{k=0}^{n-1}(-i)^k C_k z^{-3k/2} + \eta_n^{(1)}(\xi,\tfrac{1}{3})\right]\right.$$
$$\left. + \exp\left(-i\xi + \tfrac{1}{4}\pi i\right)\left[\sum_{k=0}^{n-1} i^k C_k z^{-3k/2} + \eta_n^{(2)}(\xi,\tfrac{1}{3})\right]\right\}$$
$$\left(-\frac{2\pi}{3} + \delta' \le \mathrm{phz} \le \frac{2\pi}{3} - \delta'\right). \tag{6.40}$$

Putting $\nu = \frac{2}{3}$ in equations (6.25, 6.26) yields

$$H_{2/3}^{(1)}(\xi) = \left(\frac{2}{\pi\xi}\right)^{1/2}\exp\left(i\xi - \frac{7\pi i}{12}\right)\left\{\sum_{k=0}^{n-1} i^k A_k(\tfrac{2}{3})\xi^{-k} + \eta_n^{(1)}(\xi,\tfrac{2}{3})\right\}$$
$$(-\pi + \delta \le \mathrm{ph}\xi \le 2\pi - \delta)$$
$$= \left(\frac{3}{\pi}\right)^{1/2} z^{-3/4}\exp\left(i\tfrac{2}{3}z^{3/2} - \frac{7\pi i}{12}\right)\left\{\sum_{k=0}^{n-1} i^k D_k z^{-3k/2} + \eta_n^{(1)}(\tfrac{2}{3}z^{3/2},\tfrac{2}{3})\right\}$$
$$\left(-\frac{2\pi}{3} + \delta' \le \mathrm{phz} \le \frac{4\pi}{3} - \delta'\right) \tag{6.41}$$

and

$$H_{2/3}^{(2)}(\xi) = \left(\frac{2}{\pi\xi}\right)^{1/2}\exp\left(-i\xi + \frac{7\pi i}{12}\right)\left\{\sum_{k=0}^{n-1}(-i)^k A_k(\tfrac{2}{3})\xi^{-k} + \eta_n^{(2)}(\xi,\tfrac{2}{3})\right\}$$
$$(-2\pi + \delta \le \mathrm{ph}\xi \le \pi - \delta)$$
$$= \left(\frac{3}{\pi}\right)^{1/2} z^{-3/4}\exp\left(-i\tfrac{2}{3}z^{3/2} + \frac{7\pi i}{12}\right)\left\{\sum_{k=0}^{n-1}(-i)^k D_k z^{-3k/2} + \eta_n^{(2)}(\tfrac{2}{3}z^{3/2},\tfrac{2}{3})\right\}$$
$$\left(-\frac{4\pi}{3} + \delta' \le \mathrm{phz} \le \frac{2\pi}{3} - \delta'\right). \tag{6.42}$$

The coefficients $A_k(\frac{2}{3})$ and D_k are given by

$$A_o\left(\frac{2}{3}\right) = D_o = 1 \tag{6.43}$$

and

$$A_k\left(\frac{2}{3}\right) = \frac{2^k}{3^k} D_k$$

$$= (-1)^k \frac{[9 - 16][81 - 16]\cdots[9(2k - 1)^2 - 16]}{3^{2k}\, 2^{3k}\, k!} \quad (k \geq 1) . \tag{6.44}$$

The error terms $\eta_n^{(1)}\left(\xi, \frac{2}{3}\right)$ and $\eta_n^{(2)}\left(\xi, \frac{2}{3}\right)$ are bounded by

$$|\eta_n^{(1)}\left(\xi, \frac{2}{3}\right)| \leq \frac{2^{n+1}}{3^n} |D_n| \phi_n^{(1)}(\xi) \exp\left\{\frac{7}{36}\phi_1^{(1)}(\xi)\right\} \tag{6.45}$$

and

$$|\eta_n^{(2)}\left(\xi, \frac{2}{3}\right)| \leq \frac{2^{n+1}}{3^n} |D_n| \phi_n^{(2)}(\xi) \exp\left\{\frac{7}{36}\phi_1^{(2)}(\xi)\right\} . \tag{6.46}$$

Combining the above results with equation (6.23) yields the asymptotic expansion

$$\mathrm{Ai}'(-z) = -\frac{1}{2}\pi^{-1/2}\, z^{1/4} \left\{ \exp\left(i\xi + \frac{\pi i}{4}\right)\left[\sum_{k=0}^{n-1} i^k D_k z^{-3k/2} + \eta_n^{(1)}\left(\xi, \frac{2}{3}\right)\right] \right.$$

$$\left. + \exp\left(-i\xi - \frac{\pi i}{4}\right)\left[\sum_{k=0}^{n-1} (-i)^k D_k z^{-3k/2} + \eta_n^{(2)}\left(\xi, \frac{2}{3}\right)\right] \right\}$$

$$\left(-\frac{2\pi}{3} + \delta' \leq \mathrm{phz} \leq \frac{2\pi}{3} - \delta'\right). \tag{6.47}$$

In order for asymptotic expansions of Ai(z) and Ai'(z) valid for other values of phz to be obtained, the equations [30]

$$\mathrm{Ai}(z) = \frac{1}{\pi}\left(\frac{z}{3}\right)^{1/2} K_{1/3}(\xi) \tag{6.48}$$

and

$$\mathrm{Ai}'(z) = -\frac{1}{\pi}\left(\frac{z}{\sqrt{3}}\right) K_{2/3}(\xi), \tag{6.49}$$

relating the Airy functions to the modified Bessel functions, are used. In equations (6.48, 6.49), ξ is again related to z by

$$\xi = \frac{2}{3}z^{3/2} \quad . \tag{6.50}$$

The modified Bessel functions have the asymptotic expansion [33-35]

$$K_\nu(\xi) = \left(\frac{\pi}{2\xi}\right)^{1/2} e^{-\xi}\left\{\sum_{k=0}^{n-1} A_k(\nu)\xi^{-k} + \gamma_n(\xi,\nu)\right\}$$

$$\left(-\frac{3\pi}{2} + \delta \le \mathrm{ph}\xi \le \frac{3\pi}{2} - \delta\right) , \tag{6.51}$$

where the coefficients $A_k(\nu)$ are again given by equations (6.27, 6.28). The error term $\gamma_n(\xi,\nu)$ is bounded by

$$|\gamma_n(\xi,\nu)| \le 2|A_n(\nu)|\psi_n(\xi)\exp\left\{|\nu^2 - \tfrac{1}{4}|\psi_1(\xi)\right\} , \tag{6.52}$$

where the error control function $\psi_n(\xi)$ has the bounds

$$\psi_n(\xi) \le \begin{cases} |\xi|^{-n} & \left(|\mathrm{ph}\xi| \le \frac{\pi}{2}\right) \\ \chi(n)|\xi|^{-n} & \left(\frac{\pi}{2} \le |\mathrm{ph}\xi| \le \pi\right) \\ 2\chi(n)|\mathrm{Re}(\xi)|^{-n} & \left(\pi \le |\mathrm{ph}\xi| \le \frac{3\pi}{2} - \delta\right) . \end{cases} \tag{6.53}$$

When $\nu = \frac{1}{3}$, equations (6.51, 6.52) become

$$K_{1/3}(\xi) = \left(\frac{\pi}{2\xi}\right)^{1/2} e^{-\xi}\left\{\sum_{k=0}^{n-1} A_k\left(\tfrac{1}{3}\right)\xi^{-k} + \gamma_n\left(\xi, \tfrac{1}{3}\right)\right\}$$

$$\left(-\frac{3\pi}{2} + \delta \le \mathrm{ph}\xi \le \frac{3\pi}{2} - \delta\right)$$

$$= \left(\frac{3\pi}{4}\right)^{1/2} z^{-3/4}\exp\left(-\frac{2}{3}\,z^{3/2}\right)\left\{\sum_{k=0}^{n-1}(-1)^k C_k\, z^{-3k/2} + \gamma_n\left(\frac{2}{3}\,z^{3/2}, \tfrac{1}{3}\right)\right\}$$

$$(-\pi + \delta' \le \mathrm{ph}z \le \pi - \delta') , \tag{6.54}$$

where the error term is bounded by

$$|\gamma_n(\xi, \tfrac{1}{3})| \leq \frac{2^{n+1}}{3^n} C_n \psi_n(\xi) \exp\left\{\frac{5}{36}\psi_1(\xi)\right\} . \tag{6.55}$$

Substituting equation (6.54) into equation (6.48) yields the asymptotic expansion

$$\mathrm{Ai}(z) = \tfrac{1}{2}\pi^{-1/2} z^{-1/4} e^{-\xi} \left\{ \sum_{k=0}^{n-1} (-1)^k C_k z^{-3k/2} + \gamma_n(\xi, \tfrac{1}{3}) \right\}$$

$$(-\pi + \delta' \leq \mathrm{phz} \leq \pi - \delta') . \tag{6.56}$$

Putting $\nu = \frac{2}{3}$ in equations (6.51, 6.52) yields the asymptotic expansion

$$K_{2/3}(\xi) = \left(\frac{\pi}{2\xi}\right)^{1/2} e^{-\xi} \left\{ \sum_{k=0}^{n-1} A_k\left(\tfrac{2}{3}\right) \xi^{-k} + \gamma_n(\xi, \tfrac{2}{3}) \right\}$$

$$\left(-\frac{3\pi}{2} + \delta \leq \mathrm{ph}\xi \leq \frac{3\pi}{2} - \delta\right)$$

$$= \left(\frac{3\pi}{4}\right)^{1/2} z^{-3/4} \exp\left(-\tfrac{2}{3}z^{3/2}\right) \left\{ \sum_{k=0}^{n-1} D_k z^{-3k/2} + \gamma_n\left(\tfrac{2}{3}z^{3/2}, \tfrac{2}{3}\right) \right\}$$

$$(-\pi + \delta' \leq \mathrm{phz} \leq \pi - \delta'), \tag{6.57}$$

where the error term is bounded by

$$|\gamma_n(\xi, \tfrac{2}{3})| \leq \frac{2^{n+1}}{3^n} |D_n| \psi_n(\xi) \exp\left\{\frac{7}{36}\psi_1(\xi)\right\} . \tag{6.58}$$

Combining equations (6.57) and (6.49) yields

$$\mathrm{Ai}'(z) = -\tfrac{1}{2}\pi^{-1/2} z^{1/4} e^{-\xi} \left\{ \sum_{k=0}^{n-1} D_k z^{-3k/2} + \gamma_n(\xi, \tfrac{2}{3}) \right\}$$

$$(-\pi + \delta \leq \mathrm{phz} \leq \pi - \delta) . \tag{6.59}$$

Consider the asymptotic expansion (6.40)

$$\mathrm{Ai}(-z) = \tfrac{1}{2}\pi^{-1/2}z^{-1/4}\left\{\exp\left(i\xi - \frac{\pi i}{4}\right)\left[\sum_{k=0}^{n-1}(-i)^k C_k z^{-3k/2} + \eta_n^{(1)}\left(\xi, \tfrac{1}{3}\right)\right]\right.$$
$$\left. + \exp\left(-i\xi + \frac{\pi i}{4}\right)\left[\sum_{k=0}^{n-1} i^k C_k z^{-3k/2} + \eta_n^{(2)}\left(\xi, \tfrac{1}{3}\right)\right]\right\}$$
$$\left(-\frac{2\pi}{3} + \delta \le \mathrm{phz} \le \frac{2\pi}{3} - \delta\right). \qquad (6.60)$$

Let R_1 be the ratio of the bound for the error term $\eta_n^{(1)}\left(\xi, \frac{1}{3}\right)$ to the modulus of the first neglected term in the series $S_1 = \Sigma(-i)^k C_k z^{-3k/2}$. When $0 \le \mathrm{phz} \le \frac{2\pi}{3} - \delta$, this ratio is approximately 2. When $-\frac{\pi}{3} \le \mathrm{phz} \le 0$, R_1 is approximately $2\chi(n)$. When $-\frac{2\pi}{3} + \delta \le \mathrm{phz} \le -\frac{\pi}{3}$, R_1 is approximately $4\chi(n)\csc^n\left(\frac{3}{2}\delta\right)$. As $\delta \to 0$, this last bound grows sharply, warning that if $\eta_n^{(1)}\left(\xi, \frac{1}{3}\right)$ is neglected, this asymptotic expansion for Ai(-z) is inaccurate for numerical work near the boundary $\mathrm{phz} = -\frac{2\pi}{3}$. Let R_2 be the ratio of the bound for the error term $\eta_n^{(2)}\left(\xi, \frac{1}{3}\right)$ to the modulus of the first neglected term in the series $S_2 = \Sigma i^k C_k z^{-3k/2}$. When $-\frac{2\pi}{3} + \delta \le \mathrm{phz} \le 0$, the ratio R_2 is approximately 2. When $0 \le \mathrm{phz} \le \frac{\pi}{3}$, the ratio R_2 is approximately $2\chi(n)$. When $\frac{\pi}{3} \le \mathrm{phz} \le \frac{2\pi}{3} - \delta$, the ratio R_2 is approximately $4\chi(n)\csc^n\left(\frac{3}{2}\delta\right)$. Again, as $\delta \to 0$, this bound grows sharply, indicating that the asymptotic expansion (6.60) is inaccurate for numerical work near the boundary $\mathrm{phz} = \frac{2\pi}{3}$. Therefore, the asymptotic expansion (6.60) for Ai(-z) is satisfactory for numerical computation in the phase range $-\frac{\pi}{3} \le \mathrm{phz} \le \frac{\pi}{3}$. An identical argument gives the same result for the asymptotic expansion (6.47) for Ai'(-z).

Now consider the asymptotic expansion (6.56)

$$\mathrm{Ai}(z) = \tfrac{1}{2}\pi^{-1/2}z^{-1/4}e^{-\xi}\left\{\sum_{k=0}^{n-1}(-1)^k C_k z^{-3k/2} + \gamma_n\left(\xi, \tfrac{1}{3}\right)\right\}$$
$$(-\pi + \delta \le \mathrm{phz} \le \pi - \delta) \qquad (6.61)$$

Let R be the ratio of the bound for the error term $\gamma_n\left(\xi, \frac{1}{3}\right)$ to the modulus of the first neglected term in the series $\Sigma(-1)^k C_k z^{-3k/2}$. When $-\frac{\pi}{3} \leq \text{phz} \leq \frac{\pi}{3}$, the ratio R is approximately 2. When $-\frac{2\pi}{3} \leq \text{phz} \leq -\frac{\pi}{3}$ or $\frac{\pi}{3} \leq \text{phz} \leq \frac{2\pi}{3}$, this ratio is approximately $2\chi(n)$. Finally, when $-\pi + \delta \leq \text{phz} \leq -\frac{2\pi}{3}$ or $\frac{2\pi}{3} \leq \text{phz} \leq \pi - \delta$, this ratio is approximately $4\chi(n)\sec^n(\frac{3}{2}\text{phz})$. This last value grows sharply as $\text{phz} \to -\pi$ or $\text{phz} \to \pi$, indicating that the asymptotic expansion (6.61) for Ai(z) is inaccurate for numerical work near the boundaries phz = $-\pi$ and π. Therefore, this asymptotic expansion is satisfactory for numerical work in the sector $-\frac{2\pi}{3} \leq \text{phz} \leq \frac{2\pi}{3}$. An identical argument yields the same result for the asymptotic expansion (6.59) for Ai'(z).

In summary, the Airy function Ai(z) and its derivative have the asymptotic expansions

$$\text{Ai}(z) = \frac{1}{2}\pi^{-1/2}z^{-1/4}e^{-\xi}\left\{\sum_{k=0}^{n-1}(-1)^k C_k z^{-3k/2} + \gamma_n\left(\xi, \frac{1}{3}\right)\right\}$$
$$\left(-\frac{2\pi}{3} \leq \text{phz} \leq \frac{2\pi}{3}\right), \tag{6.62}$$

$$\text{Ai}(-z) = \frac{1}{2}\pi^{-1/2}z^{-1/4}\left\{\exp\left(i\xi - \frac{\pi i}{4}\right)\left[\sum_{k=0}^{n-1}(-i)^k C_k z^{-3k/2} + \eta_n^{(1)}\left(\xi, \frac{1}{3}\right)\right]\right.$$
$$\left. + \exp\left(-i\xi + \frac{\pi i}{4}\right)\left[\sum_{k=0}^{n-1} i^k C_k z^{-3k/2} + \eta_n^{(2)}\left(\xi, \frac{1}{3}\right)\right]\right\}$$
$$\left(-\frac{\pi}{3} \leq \text{phz} \leq \frac{\pi}{3}\right), \tag{6.63}$$

$$\text{Ai}'(z) = -\frac{1}{2}\pi^{-1/2}z^{1/4}e^{-\xi}\left\{\sum_{k=0}^{n-1} D_k z^{-3k/2} + \gamma_n\left(\xi, \frac{2}{3}\right)\right\}$$
$$\left(-\frac{2\pi}{3} \leq \text{phz} \leq \frac{2\pi}{3}\right), \tag{6.64}$$

and

$$\text{Ai}(-z) = -\frac{1}{2}\pi^{-1/2}z^{1/4}\left\{\exp\left(i\xi + \frac{\pi i}{4}\right)\left[\sum_{k=0}^{n-1} i^k D_k z^{-3k/2} + \eta_n^{(1)}\left(\xi, \frac{2}{3}\right)\right]\right.$$

$$\left. + \exp\left(-i\xi - \frac{\pi i}{4}\right)\left[\sum_{k=0}^{n-1} (-i)^k D_k z^{-3k/2} + \eta_n^{(2)}\left(\xi, \frac{2}{3}\right)\right]\right\}$$

$$\left(-\frac{\pi}{3} \leq \text{phz} \leq \frac{\pi}{3}\right), \tag{6.65}$$

where

$$\xi = \frac{2}{3}z^{3/2} . \tag{6.66}$$

The coefficients C_k and D_k are given by equations (6.37) and (6.44). They can be calculated using the recursion relations

$$C_o = 1 , \tag{6.67}$$

$$C_k = \frac{9(2k-1)^2 - 4}{48k} C_{k-1} \qquad (k \geq 1) , \tag{6.68}$$

$$D_o = 1 , \tag{6.69}$$

and

$$D_k = -\frac{9(2k-1)^2 - 16}{48k} D_{k-1} \qquad (k \geq 1) . \tag{6.70}$$

The error terms $\eta_n^{(1)}\left(\xi, \frac{1}{3}\right)$, $\eta_n^{(2)}\left(\xi, \frac{1}{3}\right)$, $\eta_n^{(1)}\left(\xi, \frac{2}{3}\right)$, $\eta_n^{(2)}\left(\xi, \frac{2}{3}\right)$, $\gamma_n\left(\xi, \frac{1}{3}\right)$, and $\gamma_n\left(\xi, \frac{2}{3}\right)$ are bounded by

$$\left|\eta_n^{(1)}\left(\xi, \frac{1}{3}\right)\right| \leq \frac{2^{n+1}}{3^n} C_n \phi_n^{(1)}(\xi) \exp\left\{\frac{5}{36}\phi_1^{(1)}(\xi)\right\}, \tag{6.71}$$

$$\left|\eta_n^{(2)}\left(\xi, \frac{1}{3}\right)\right| \leq \frac{2^{n+1}}{3^n} C_n \phi_n^{(2)}(\xi) \exp\left\{\frac{5}{36}\phi_1^{(2)}(\xi)\right\}, \tag{6.72}$$

$$|\eta_n^{(1)}\left(\xi, \frac{2}{3}\right)| \leq \frac{2^{n+1}}{3^n}|D_n|\phi_n^{(1)}(\xi)\exp\left\{\frac{7}{36}\phi_1^{(1)}(\xi)\right\}, \tag{6.73}$$

$$|\eta_n^{(2)}\left(\xi, \frac{2}{3}\right)| \leq \frac{2^{n+1}}{3^n}|D_n|\phi_n^{(2)}(\xi)\exp\left\{\frac{7}{36}\phi_1^{(2)}(\xi)\right\}, \tag{6.74}$$

$$|\gamma_n\left(\xi, \frac{1}{3}\right)| \leq \frac{2^{n+1}}{3^n}\, C_n\psi_n(\xi)\exp\left\{\frac{5}{36}\psi_1(\xi)\right\}, \tag{6.75}$$

and

$$|\gamma_n\left(\xi, \frac{2}{3}\right)| \leq \frac{2^{n+1}}{3^n}|D_n|\psi_n(\xi)\exp\left\{\frac{7}{36}\psi_1(\xi)\right\}. \tag{6.76}$$

The error control functions $\phi_n^{(1)}(\xi)$, $\phi_n^{(2)}(\xi)$, and $\psi_n(\xi)$ are bounded by

$$\phi_n^{(1)}(\xi) \leq \begin{cases} |\xi|^{-n} & (0 \leq \mathrm{ph}\xi \leq \pi) \\ \chi(n)|\xi|^{-n} & \left(-\frac{\pi}{2} \leq \mathrm{ph}\xi \leq 0\ ;\ \pi \leq \mathrm{ph}\xi \leq \frac{3\pi}{2}\right) \\ 2\chi(n)|\mathrm{Im}(\xi)|^{-n} & \left(-\pi + \delta \leq \mathrm{ph}\xi \leq -\frac{\pi}{2}\ ;\ \frac{3\pi}{2} \leq \mathrm{ph}\xi \leq 2\pi - \delta\right), \end{cases} \tag{6.77}$$

$$\phi_n^{(2)}(\xi) \leq \begin{cases} |\xi|^{-n} & (-\pi \leq \mathrm{ph}\xi \leq 0) \\ \chi(n)|\xi|^{-n} & \left(0 \leq \mathrm{ph}\xi \leq \frac{\pi}{2}\ ;\ -\frac{3}{2} \leq \mathrm{ph}\xi \leq -\pi\right) \\ 2\chi(n)|\mathrm{Im}(\xi)|^{-n} & \left(\frac{\pi}{2} \leq \mathrm{ph}\xi \leq \pi - \delta\ ;\ -2\pi + \delta \leq \mathrm{ph}\xi \leq -\frac{3\pi}{2}\right), \end{cases} \tag{6.78}$$

and

$$\psi_n(\xi) \leq \begin{cases} |\xi|^{-n} & \left(-\frac{\pi}{2} \leq \mathrm{ph}\xi \leq \frac{\pi}{2}\right) \\ \chi(n)|\xi|^{-n} & \left(-\pi \leq \mathrm{ph}\xi \leq -\frac{\pi}{2}\ ;\ \frac{\pi}{2} \leq \mathrm{ph}\xi \leq \pi\right) \\ 2\chi(n)|\mathrm{Re}(\xi)|^{-n} & \left(-\frac{3\pi}{2} + \delta \leq \mathrm{ph}\xi \leq -\pi\ ;\ \pi \leq \mathrm{ph}\xi \leq \frac{3\pi}{2} - \delta\right), \end{cases} \tag{6.79}$$

where $\delta > 0$ and

$$\chi(n) = \pi^{1/2} \frac{\Gamma\left(\frac{1}{2}n + 1\right)}{\Gamma\left(\frac{1}{2}n + \frac{1}{2}\right)} \tag{6.80}$$

Substituting the results for the asymptotic expansions of the Airy function and its derivative into equations (6.12-6.15) yields the asymptotic expansions of the modified Hankel functions of order one-third and their derivatives. The results are

$$h_1(z) = (12)^{1/6}\pi^{-1/2}z^{-1/4}\exp\left(i\xi - \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} (-i)^k C_k z^{-3k/2} + \eta_n^{(1)}\left(\xi, \frac{1}{3}\right)\right]$$

$$\left(-\frac{\pi}{3} \le \text{phz} \le \pi\right), \tag{6.81}$$

$$h_1(z) = (12)^{1/6}\pi^{-1/2}z^{-1/4}\exp\left(-i\xi - \frac{11\pi i}{12}\right)\left[\sum_{k=0}^{n-1} i^k C_k z^{-3k/2} + \eta_n^{(1)}\left(\xi e^{\pi i}, \frac{1}{3}\right)\right]$$

$$+ (12)^{1/6}\pi^{-1/2}z^{-1/4}\exp\left(i\xi - \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} (-i)^k C_k z^{-3k/2} + \eta_n^{(1)}\left(\xi e^{2\pi i}, \frac{1}{3}\right)\right]$$

$$\left(-\pi \le \text{phz} \le -\frac{\pi}{3}\right), \tag{6.82}$$

$$h_1'(z) = i(12)^{1/6}\pi^{-1/2}z^{1/4}\exp\left(i\xi - \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} i^k D_k z^{-3k/2} + \eta_n^{(1)}\left(\xi, \frac{2}{3}\right)\right]$$

$$\left(-\frac{\pi}{3} \le \text{phz} \le \pi\right), \tag{6.83}$$

$$h_1'(z) = i(12)^{1/6}\pi^{-1/2}z^{1/4}\exp\left(i\xi - \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} i^k D_k z^{-3k/2} + \eta_n^{(1)}\left(\xi e^{2\pi i}, \frac{2}{3}\right)\right]$$

$$- i(12)^{1/6}\pi^{-1/2}z^{1/4}\exp\left(-i\xi - \frac{11\pi i}{12}\right)\left[\sum_{k=0}^{n-1} (-i)^k D_k z^{-3k/2} + \eta_n^{(1)}\left(\xi e^{\pi i}, \frac{2}{3}\right)\right]$$

$$\left(-\pi \le \text{phz} \le -\frac{\pi}{3}\right), \tag{6.84}$$

$$h_2(z) = (12)^{1/6}\pi^{-1/2}z^{-1/4}\exp\left(-i\xi + \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} i^k C_k z^{-3k/2} + \eta_n^{(2)}\left(\xi, \frac{1}{3}\right)\right]$$

$$\left(-\pi \leq \text{phz} \leq \frac{\pi}{3}\right), \tag{6.85}$$

$$h_2(z) = (12)^{1/6}\pi^{-1/2}z^{-1/4}\exp\left(-i\xi + \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} i^k C_k z^{-3k/2} + \eta_n^{(2)}\left(\xi e^{-2\pi i}, \frac{1}{3}\right)\right]$$

$$+ (12)^{1/6}\pi^{-1/2}z^{-1/4}\exp\left(i\xi + \frac{11\pi i}{12}\right)\left[\sum_{k=0}^{n-1} (-i)^k C_k z^{-3k/2} + \eta_n^{(2)}\left(\xi e^{-\pi i}, \frac{1}{3}\right)\right]$$

$$\left(\frac{\pi}{3} \leq \text{phz} \leq \pi\right), \tag{6.86}$$

$$h_2'(z) = -i(12)^{1/6}\pi^{-1/2}z^{1/4}\exp\left(-i\xi + \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} (-i)^k D_k z^{-3k/2} + \eta_n^{(2)}\left(\xi, \frac{2}{3}\right)\right]$$

$$\left(-\pi \leq \text{phz} \leq \frac{\pi}{3}\right), \tag{6.87}$$

and

$$h_2'(z) = i(12)^{1/6}\pi^{-1/2}z^{1/4}\exp\left(i\xi + \frac{11\pi i}{12}\right)\left[\sum_{k=0}^{n-1} i^k D_k z^{-3k/2} + \eta_n^{(2)}\left(\xi e^{-\pi i}, \frac{2}{3}\right)\right]$$

$$- i(12)^{1/6}\pi^{-1/2}z^{1/4}\exp\left(-i\xi + \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} (-i)^k D_k z^{-3k/2} + \eta_n^{(2)}\left(\xi e^{-2\pi i}, \frac{2}{3}\right)\right]$$

$$\left(\frac{\pi}{3} \leq \text{phz} \leq \pi\right) \tag{6.88}$$

Using equations (6.17-6.20), which relate the functions $k_1(z)$ and $k_2(z)$ to the Airy functions, together with the asymptotic expansions for the Airy functions yields

$$k_1(z) = (12)^{1/6}\pi^{-1/2}z^{-1/4}\exp\left(i\xi - \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1}(-i)^k C_k z^{-3k/2} + \eta_n^{(1)}\left(\xi, \frac{1}{3}\right)\right]$$
$$\left(-\frac{\pi}{3} \le \text{phz} \le \pi\right), \tag{6.89}$$

$$k_1(z) = (12)^{1/6}\pi^{-1/2}z^{-1/4}\exp\left(-i\xi - \frac{11\pi i}{12}\right)\left[\sum_{k=0}^{n-1} i^k C_k z^{-3k/2} + \eta_n^{(1)}\left(\xi e^{\pi i}, \frac{1}{3}\right)\right]$$
$$+ (12)^{1/6}\pi^{-1/2}z^{-1/4}\exp\left(i\xi - \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1}(-i)^k C_k z^{-3k/2} + \eta_n^{(1)}\left(\xi e^{2\pi i}, \frac{1}{3}\right)\right]$$
$$\left(-\pi \le \text{phz} \le -\frac{\pi}{3}\right), \tag{6.90}$$

$$k_1'(z) = i(12)^{1/6}\pi^{-1/2}z^{1/4}\exp\left(i\xi - \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} i^k D_k z^{-3k/2} + \eta_n^{(1)}\left(\xi, \frac{2}{3}\right)\right]$$
$$\left(-\frac{\pi}{3} \le \text{phz} \le \pi\right), \tag{6.91}$$

$$k_1'(z) = i(12)^{1/6}\pi^{-1/2}z^{1/4}\exp\left(i\xi - \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} i^k D_k z^{-3k/2} + \eta_n^{(1)}\left(\xi e^{2\pi i}, \frac{2}{3}\right)\right]$$
$$- i(12)^{1/6}\pi^{-1/2}z^{1/4}\exp\left(-i\xi - \frac{11\pi i}{12}\right)\left[\sum_{k=0}^{n-1}(-i)^k D_k z^{-3k/2} + \eta_n^{(2)}\left(\xi e^{\pi i}, \frac{2}{3}\right)\right]$$
$$\left(-\pi \le \text{phz} \le -\frac{\pi}{3}\right), \tag{6.92}$$

$$k_2(z) = (12)^{1/6}\pi^{-1/2}z^{-1/4}\exp\left(-i\xi + \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} i^k C_k z^{-3k/2} + \eta_n^{(2)}\left(\xi, \frac{1}{3}\right)\right]$$
$$- (12)^{1/6}\pi^{-1/2}z^{-1/4}\exp\left(i\xi + \frac{11\pi i}{12}\right)\left[\sum_{k=0}^{n-1}(-i)^k C_k z^{-3k/2} + \eta_n^{(1)}\left(\xi, \frac{1}{3}\right)\right]$$
$$\left(-\frac{\pi}{3} \le \text{phz} \le \frac{\pi}{3}\right), \tag{6.93}$$

$$k_2(z) = -(12)^{1/6}\pi^{-1/2}z^{-1/4}\exp\left(i\xi + \frac{11\pi i}{12}\right)\left[\sum_{k=0}^{n-1} (-i)^k C_k z^{-3k/2} + \gamma_n\left(\xi e^{3\pi i/2}, \frac{1}{3}\right)\right]$$

$$\left(-\frac{5\pi}{3} \leq \text{phz} \leq -\frac{\pi}{3}\right), \tag{6.94}$$

$$k_2(z) = (12)^{1/6}\pi^{-1/2}z^{-1/4}\exp\left(-i\xi + \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} i^k C_k z^{-3k/2} + \gamma_n\left(\xi e^{-3\pi i/2}, \frac{1}{3}\right)\right]$$

$$\left(\frac{\pi}{3} \leq \text{phz} \leq \frac{5\pi}{3}\right), \tag{6.95}$$

$$k_2'(z) = -i(12)^{1/6}\pi^{-1/2}z^{1/4}\exp\left(i\xi + \frac{11\pi i}{12}\right)\left[\sum_{k=0}^{n-1} i^k D_k z^{-3k/2} + \eta_n^{(1)}\left(\xi, \frac{2}{3}\right)\right]$$

$$- i(12)^{1/6}\pi^{-1/2}z^{1/4}\exp\left(-i\xi + \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} (-i)^k D_k z^{-3k/2} + \eta_n^{(2)}\left(\xi, \frac{2}{3}\right)\right]$$

$$\left(-\frac{\pi}{3} \leq \text{phz} \leq \frac{\pi}{3}\right), \tag{6.96}$$

$$k_2'(z) = -i(12)^{1/6}\pi^{-1/2}z^{1/4}\exp\left(i\xi + \frac{11\pi i}{12}\right)\left[\sum_{k=0}^{n-1} i^k D_k z^{-3k/2} + \gamma_n\left(\xi e^{3\pi i/2}, \frac{2}{3}\right)\right]$$

$$\left(-\frac{5\pi}{3} \leq \text{phz} \leq -\frac{\pi}{3}\right), \tag{6.97}$$

and

$$k_2'(z) = -i(12)^{1/6}\pi^{-1/2}z^{1/4}\exp\left(-i\xi + \frac{5\pi i}{12}\right)\left[\sum_{k=0}^{n-1} (-i)^k D_k z^{-3k/2} + \gamma_n\left(\xi e^{-3\pi i/2}, \frac{2}{3}\right)\right]$$

$$\left(\frac{\pi}{3} \leq \text{phz} \leq \frac{5\pi}{3}\right). \tag{6.98}$$

Equations (6.94) and (6.95) are equivalent, as are equations (6.97) and (6.98). This is to be expected since the Airy function and its derivative are entire.

The preceding asymptotic expansions can be used to calculate Ai(z) and Ai'(z) when |z| is large. The given error bounds can be used to determine how large |z| must be in order to guarantee a given accuracy. For small values of |z| the Airy functions can be evaluated by using the series expansions [30]

$$\mathrm{Ai}(z) = c_1 f(z) - c_2 g(z) \tag{6.99}$$

and

$$\mathrm{Ai}'(z) = c_1 f'(z) - c_2 g'(z) , \tag{6.100}$$

where

$$f(z) = \sum_{k=0}^{\infty} 3^k \frac{\Gamma\left(k + \frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \frac{z^{3k}}{(3k)!} , \tag{6.101}$$

$$f'(z) = \sum_{k=1}^{\infty} 3^k \frac{\Gamma\left(k + \frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \frac{z^{3k-1}}{(3k-1)!} , \tag{6.102}$$

$$g(z) = \sum_{k=0}^{\infty} 3^k \frac{\Gamma\left(k + \frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \frac{z^{3k+1}}{(3k+1)!} , \tag{6.103}$$

and

$$g'(z) = \sum_{k=0}^{\infty} 3^k \frac{\Gamma\left(k + \frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \frac{z^{3k}}{(3k)!} \tag{6.104}$$

The constants c_1 and c_2 are

$$c_1 = \frac{1}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} \tag{6.105}$$

and

$$c_2 = \frac{1}{3^{1/3}\Gamma\left(\frac{1}{3}\right)} \tag{6.106}$$

In this section an upper bound for the error introduced by truncating the above series for f(z) and g(z) and their derivatives at N terms will be obtained. The largest absolute error due to truncation will occur when all the terms in the series have the same sign, i.e., when z = x is real and positive. Except in the neighborhood of the zeros of Ai(z) and Ai'(z), the largest relative error will also occur when z = x is real and positive.

The truncated series approximations for the Airy function and its derivative are given by

$$Ai_N(z) = c_1 f_N(z) - c_2 g_N(z) \tag{6.107}$$

and

$$Ai'_N(z) = c_1 f'_N(z) - c_2 g'_N(z) \ , \tag{6.108}$$

where

$$f_N(z) = \sum_{k=0}^{N-1} 3^k \frac{\Gamma\left(k + \frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \frac{z^{3k}}{(3k)!} \ , \tag{6.109}$$

$$f'_N(z) = \sum_{k=1}^{N-1} 3^k \frac{\Gamma\left(k + \frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \frac{z^{3k-1}}{(3k - 1)!} \ , \tag{6.110}$$

$$g_N(z) = \sum_{k=0}^{N-1} 3^k \frac{\Gamma\left(k + \frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \frac{z^{3k+1}}{(3k + 1)!} \ , \tag{6.111}$$

and

$$g'_N(z) = \sum_{k=0}^{N-1} 3^k \frac{\Gamma\left(k + \frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} \frac{z^{3k}}{(3k)!} \tag{6.112}$$

are N term approximations to the infinite series (6.101-6.104).

The absolute error in approximating Ai(z) with $Ai_N(z)$ is $|Ai(z) - Ai_N(z)|$. This is bounded by

$$|Ai(z) - Ai_N(z)| \leq c_1|\varepsilon_N(z)| + c_2|\delta_N(z)| \ , \tag{6.113}$$

where

$$\varepsilon_N(z) = f(z) - f_N(z) \tag{6.114}$$

and

$$\delta_N(z) = g(z) - g_N(z) \ . \tag{6.115}$$

Similarly, for the derivative Ai'(z)

$$|Ai'(z) - Ai'_N(z)| \leq c_1|\bar{\varepsilon}_N(z)| + c_2|\bar{\delta}_N(z)| \ , \tag{6.116}$$

where

$$\bar{\varepsilon}_N(z) = f'(z) - f'_N(z) \tag{6.117}$$

and

$$\bar{\delta}_N(z) = g'(z) - g'_N(z) \ . \tag{6.118}$$

An upper bound for $|\varepsilon_N(z)|$ can be obtained in the following manner. The truncation error $\varepsilon_N(z)$ for f(z) is

$$\varepsilon_N(z) = \sum_{k=N}^{\infty} 3^k \frac{\Gamma\left(k + \frac{1}{3}\right)}{\Gamma\left(\frac{1}{3}\right)} \frac{z^{3k}}{(3k)!}$$

$$= \frac{3^N z^{3N}}{\Gamma\left(\frac{1}{3}\right)} \sum_{j=0}^{\infty} 3^k \frac{\Gamma\left(j + N + \frac{1}{3}\right)}{\Gamma(3j + 3N + 1)} 3^j z^{3j} \ . \tag{6.119}$$

The triplication formula for the gamma function [36] yields

$$\Gamma(3j + 3N + 1) = \Gamma\left(3\left[j + N + \frac{1}{3}\right]\right)$$

$$= \frac{1}{2\pi} 3^{3j+3N+\frac{1}{2}} \Gamma\left(j + N + \frac{1}{3}\right)\Gamma\left(j + N + \frac{2}{3}\right)\Gamma\left(j + N + 1\right) \ . \tag{6.120}$$

Therefore

$$\varepsilon_N(z) = \frac{2\pi z^{3N}}{\Gamma\left(\frac{1}{3}\right)3^{2N+\frac{1}{2}}} \sum_{j=0}^{\infty} \frac{1}{\Gamma\left(j + N + \frac{2}{3}\right)\Gamma(j + N + 1)} \left(\frac{z^3}{3^2}\right)^j \ . \tag{6.121}$$

Now,

$$\Gamma\left(j + N + \frac{2}{3}\right) > \Gamma(j + N) \tag{6.122}$$

for all $j \geq 0$ if $N \geq 2$. Hence

$$|\varepsilon_N(z)| < \frac{2\pi|z|^{3N}}{\Gamma\left(\frac{1}{3}\right)3^{2N+\frac{1}{2}}} \sum_{j=0}^{\infty} \frac{1}{\Gamma(j + N)\,\Gamma(j + N + 1)} \left(\frac{|z|^3}{9}\right)^j \quad (N \geq 2). \tag{6.123}$$

The gamma functions in the sum in (6.123) satisfy the inequalities

$$\Gamma(j + N) \geq \Gamma(N)\Gamma(j + 1) = (N - 1)!j! \tag{6.124}$$

and

$$\Gamma(j + N + 1) \geq \Gamma(N + 1)\Gamma(j + 1) = N!j! \ . \tag{6.125}$$

Therefore,

$$|\varepsilon_N(z)| < \frac{2\pi|z|^{3N}}{\Gamma\left(\frac{1}{3}\right)3^{2N+\frac{1}{2}}(N - 1)!N!} \sum_{j=0}^{\infty} \frac{1}{(j!)^2} \left(\frac{|z|^3}{9}\right)^j \ . \tag{6.126}$$

The modified Bessel function

$$I_o\left(\frac{2}{3}|z|^{3/2}\right) = \sum_{j=0}^{\infty} \frac{1}{(j!)^2}\left(\frac{|z|^3}{9}\right)^j \tag{6.127}$$

satisfies the inequality [15]

$$I_o\left(\frac{2}{3}|z|^{3/2}\right) \leq \exp\left(\frac{2}{3}|z|^{3/2}\right) \ . \tag{6.128}$$

This yields

$$|\varepsilon_N(z)| < \frac{2\pi}{3^{1/2}\Gamma\left(\frac{1}{3}\right)(N - 1)!N!} \left(\frac{|z|^3}{9}\right)^N \exp\left(\frac{2}{3}|z|^{3/2}\right) \quad (N \geq 2) \ , \tag{6.129}$$

or

$$c_1|\varepsilon_N(z)| < \frac{1}{3^{2/3}(N - 1)!N!} \left(\frac{|z|^3}{9}\right)^N \exp\left(\frac{2}{3}|z|^{3/2}\right) \quad (N \geq 2). \tag{6.130}$$

Similarly, the truncation errors $\delta_N(z)$, $\bar{\varepsilon}_N(z)$, and $\bar{\delta}_N(z)$ are bounded by

$$|\delta_N(z)| < \frac{2\pi\sqrt{3}}{\Gamma\left(\frac{2}{3}\right)(N!)^2}|z|\left(\frac{|z|^3}{9}\right)^N \exp\left(\frac{2}{3}|z|^{3/2}\right) \qquad (N \geq 1) , \qquad (6.131)$$

$$|\bar{\varepsilon}_N(z)| < \frac{2\pi\sqrt{3}}{\Gamma\left(\frac{1}{3}\right)[(N-1)!]^2}\frac{1}{|z|}\left(\frac{|z|^3}{9}\right)^N \exp\left(\frac{2}{3}|z|^{3/2}\right) \qquad (N \geq 2) , \qquad (6.132)$$

and

$$|\bar{\delta}_N(z)| < \frac{2\pi}{\sqrt{3}\,\Gamma\left(\frac{2}{3}\right)(N-1)!N!}\left(\frac{|z|^3}{9}\right)^N \exp\left(\frac{2}{3}|z|^{3/2}\right) \qquad (N \geq 2) . \qquad (6.133)$$

The absolute errors due to truncation of the infinite series (6.101-6.104) for Ai(z) and Ai'(z) are therefore bounded by

$$|Ai(z) - Ai_N(z)| < \frac{1}{3^{2/3}(N-1)!N!}\left(\frac{|z|^3}{9}\right)^N \exp\left(\frac{2}{3}|z|^{3/2}\right)\left[1 + 3^{-2/3}\,\frac{|z|}{N}\right]$$

$$(N \geq 2) , \qquad (6.134)$$

and

$$|Ai'(z) - Ai'_N(z)| < \frac{3^{1/3}}{[(N-1)!]^2}\frac{1}{|z|}\left(\frac{|z|^3}{9}\right)^N \exp\left(\frac{2}{3}|z|^{3/2}\right)\left[1 + 3^{-2/3}\,\frac{|z|}{N}\right]$$

$$(N \geq 2) . \qquad (6.135)$$

When $z = x$ is real and positive, the asymptotic approximations for Ai(x) and Ai'(x) can be used easily to obtain bounds for the relative error. The Airy function Ai(x) and its derivative have the asymptotic approximations

$$Ai(x) = \tfrac{1}{2}\pi^{-1/2}\,x^{-1/4}\,\exp\left(-\tfrac{2}{3}x^{3/2}\right)\left[1 + \gamma_1\left(\tfrac{2}{3}x^{3/2}, \tfrac{1}{3}\right)\right] \qquad (x > 0) \quad (6.136)$$

and

$$Ai'(x) = -\tfrac{1}{2}\pi^{-1/2}\,x^{1/4}\,\exp\left(-\tfrac{2}{3}x^{3/2}\right)\left[1 + \gamma_1\left(\tfrac{2}{3}x^{3/2}, \tfrac{2}{3}\right)\right] \quad (x > 0) , \quad (6.137)$$

where the error terms are bounded by

$$|\gamma_1\left(\frac{2}{3}x^{3/2}\ ,\ \frac{1}{3}\right)| \leq \frac{5}{24}x^{-3/2}\exp\left(\frac{5}{24}x^{-3/2}\right) \tag{6.138}$$

and

$$|\gamma_1\left(\frac{2}{3}x^{3/2}\ ,\ \frac{2}{3}\right)| \leq \frac{7}{24}x^{-3/2}\exp\left(\frac{7}{24}x^{-3/2}\right). \tag{6.139}$$

The relative error in the value of Ai(x) due to truncation is therefore bounded by

$$\frac{|Ai(x) - Ai_N(x)|}{|Ai(x)|} < \frac{2\sqrt{\pi}x^{1/4}}{3^{2/3}(N-1)!N!}\left(\frac{x^3}{9}\right)^N\exp\left(\frac{4}{3}x^{3/2}\right)\left[1 + 3^{-2/3}\,\frac{x}{N}\right]\left[1 - \frac{5}{24}x^{-3/2}\exp\left(\frac{5}{24}x^{-3/2}\right)\right]^{-1} \quad (x > 0.52,\ N \geq 2), \tag{6.140}$$

while the relative error in Ai'(x) due to truncation is bounded by

$$\frac{|Ai'(x) - Ai'_N(x)|}{|Ai'(x)|} < \frac{2\sqrt{\pi}\ 3^{1/3}x^{-5/4}}{[(N-1)!]^2}\left(\frac{x^3}{9}\right)^N\exp\left(\frac{4}{3}x^{3/2}\right)\left[1 + 3^{-2/3}\,\frac{x}{N}\right]\left[1 - \frac{7}{24}x^{-3/2}\exp\left(\frac{7}{24}x^{-3/2}\right)\right]^{-1} \quad (x > 0.65,\ N \geq 2)\ . \tag{6.141}$$

These bounds for the absolute and relative errors in Ai(x) and Ai'(x) due to truncation are upper bounds. In most cases these bounds are unduly pessimistic.

Another major source of error in calculating the Airy function and its derivative from equations (6.99, 6.100) is cancellation error. The functions f(z) and g(z) can be expressed in terms of the Airy functions Ai(z) and Bi(z) by [30]

$$f(z) = \frac{1}{2c_1}\left[\frac{1}{\sqrt{3}}Bi(z) + Ai(z)\right] \tag{6.142}$$

and

$$g(z) = \frac{1}{2c_2}\left[\frac{1}{\sqrt{3}}Bi(z) - Ai(z)\right]\ . \tag{6.143}$$

The derivatives of f(z) and g(z) are given by

$$f'(z) = \frac{1}{2c_1}\left[\frac{1}{\sqrt{3}}Bi'(z) + Ai'(z)\right] \tag{6.144}$$

and

$$g'(z) = \frac{1}{2c_2}\left[\frac{1}{\sqrt{3}}Bi'(z) + Ai'(z)\right]. \tag{6.145}$$

When z = x is real and positive, the Airy functions Ai(x) and Bi(x) and their derivatives have the asymptotic approximations [30]

$$Ai(x) \sim \frac{1}{2}\pi^{-1/2}x^{-1/4}e^{-\xi}\ , \tag{6.146}$$

$$Bi(x) \sim \pi^{-1/2}x^{-1/4}e^{\xi}\ , \tag{6.147}$$

$$Ai'(x) \sim -\frac{1}{2}\pi^{-1/2}x^{1/4}e^{-\xi}\ , \tag{6.148}$$

and

$$Bi'(x) \sim \pi^{-1/2}x^{1/4}e^{\xi}\ , \tag{6.149}$$

where

$$\xi = \frac{2}{3}x^{3/2}\ . \tag{6.150}$$

Thus, the functions f(x) and g(x) and their derivatives all increase exponentially with increasing x. Since Ai(x) and Ai'(x) decrease exponentially with increasing x, severe cancellation errors can arise when using equations (6.99, 6.100) to calculate Ai(x) and Ai'(x). For example, at x = 4.0 the value of Ai(x) is Ai(x) $\cong$ 0.000952, while the values of $c_1f(x)$ and $c_2g(x)$ are $c_1f(x) \cong 24.204$ and $c_2g(x) \cong 24.204$. In order to obtain three significant figures for Ai(x) at x = 4.0, eight significant figures are needed in $c_1f(x)$ and $c_2g(x)$. For reasonably large values of N, cancellation error is much more significant than truncation error.

7. EXTENDED COMPLEX ARITHMETIC

Because of the exponential nature of the Airy functions, the numerical evaluation of the modal function (2.124) easily can yield complex numbers with magnitudes as large as 10^{+1000} or as small as 10^{-1000}. Numbers of this size are outside the range of most computers, so special methods to handle them had to be incorporated into the waveguide program. A complex amplitude, ξ, and a real exponent, ϕ, were associated with each complex number z, i.e.,

$$z = \xi e^{\phi} . \tag{7.1}$$

No attempt was made to normalize the complex amplitudes ξ in XWVG. The complex amplitudes ξ typically have a magnitude falling between 10^{+3} and 10^{-3}. By handling the complex amplitude and real exponent separately, the very large and very small numbers obtained when evaluating the modal function over the search region of the complex q_{11}-plane are easily handled.

If $z_1 = \xi_1 e^{\phi_1}$ and $z_2 = \xi_2 e^{\phi_2}$ are two extended complex numbers, their product is the extended complex number

$$z = z_1 z_2 = \xi e^{\phi} , \tag{7.2}$$

where

$$\xi = \xi_1 \xi_2 \tag{7.3}$$

and

$$\phi = \phi_1 + \phi_2 . \tag{7.3}$$

If $\phi_1 \geq \phi_2$, the sum of the two extended complex numbers z_1 and z_2 is the extended complex number

$$z = z_1 + z_2 = \xi e^{\phi} , \tag{7.4}$$

where

$$\xi = \xi_1 + \xi_2 \exp(\phi_2 - \phi_1) \tag{7.5}$$

and

$$\phi = \phi_1 . \tag{7.6}$$

If $\phi_2 > \phi_1$, the complex amplitude and the real exponent for the sum would equal

$$\xi = \xi_2 + \xi_1 \exp(\phi_1 - \phi_2) \tag{7.7}$$

and

$$\phi = \phi_2 \ . \tag{7.8}$$

8. DESCRIPTION OF XWVG PROGRAM ELEMENTS

This section describes the 20 program elements comprising XWVG and gives the relationship between the different program elements.

MAIN

Description: MAIN is the controlling program element for the trilinear modified refractivity profile tropospheric propagation program XWVG. Besides calling the appropriate subroutines to determine the modal eigenvalues and evaluate the coherent and incoherent mode sums for the field strength, MAIN also performs all data input and output, sets up the search rectangles for the Shellman-Morfitt root-finding routine, and calculates the "constants" needed by other program elements.

The data for MAIN is read in from logical file number 13, the read file. Logical file number 13 is assumed to have one data item per record, with the data items in the following order:

MFILE - If MFILE = 0 calculate eigenvalues. If MFILE = 1 read eigenvalues from logical file number 12.

MPOL - If MPOL = 0 horizontal polarization. If MPOL = 1 vertical polarization.

FREQ - The frequency, in MHz, of the propagating electromagnetic wave.

ALPHA1 - The slope, in units of 1/meter, of the first (lowest) layer in the trilinear model of the troposphere. The α_i's are related to $\frac{dM_i}{dz}$ by equation (2.13). ALPHA1 must be positive.

ALPHA2 - The slope of the second (middle) layer in the trilinear model of the troposphere. ALPHA2 may be positive or negative but may not be zero.

ALPHA3 - The slope of the third (top) layer in the trilinear model of the troposphere. ALPHA3 must be positive. For the standard atmosphere $\alpha = 0.236 \times 10^{-6}$/m.

Z2 - The height, in meters, of the interface between the first and second layers of the trilinear model of the troposphere. The ground is assumed to be located at $z = 0$.

Z3 - The height, in meters, of the interface between the second and third layers of the trilinear model of the troposphere. Z3 must be greater than Z2.

ZREF - The height, in meters, at which the value of the modified refractivity, M(z), is specified. ZREF is assumed to be in the first layer, i.e.,

$$0 \leq \text{ZREF} \leq \text{Z2}.$$

REFM - The value of the modified refractivity, in M-units, at the reference height ZREF.

ETA - The absorptivity of the tropospheric gases.

TEMP - The temperature of the seawater, in degrees Celsius, over the propagation path.

SALT - The salinity of the seawater, in grams of salt per kilogram of seawater, over the propagation path.

DELTA - The root mean square bump height, in meters, of the ocean surface over the propagation path.

ALOSS - The maximum loss rate, in dB/km, of the modes which will be found.

RNGINT - The initial range separation, in kilometers, between the transmitter and receiver.

NRNG - The number of range values for which field strength values will be calculated.

RNGSTP - The increment, in kilometers, by which the range changes.

ZRINIT - The initial height, in meters, of the receiver.

NZRCVR - The number of receiver heights for which the field strength will be calculated.

ZRSTEP - The increment, in meters, by which the receiver height changes.

ZXINIT - The initial height, in meters, of the transmitter.

NZXMTR - The number of transmitter heights for which the field strength will be calculated.

ZXSTEP - The increment, in meters, by which the transmitter height changes.

If the modal eigenvalues for a particular model have already been calculated and are stored in logical file number 12, then MFILE can be set equal to one and the modal eigenvalues will not be calculated again. This can save considerable computer time and expense. If the modal eigenvalues are to be calculated then set MFILE = 0. After the modal eigenvalues are calculated, MAIN writes their value in logical file number 12 so they can be stored for future use. MAIN writes the output of the trilinear duct tropospheric program to locial file number 11, the write file.

MAIN evaluates many of the quantities depending only upon the frequency, polarization, and duct parameters which are used by other program elements. These quantities are passed to the appropriate subroutines via Common Block areas.

The search rectangles used by the Shellman-Morfitt root-finding routine are set up by MAIN. The bottom of the search rectangle is

$$[\mathrm{Im}(q_{11})]_{\mathrm{bottom}} = \mathrm{TBOT} = 0.0 \ ,$$

while the top of the search rectangle is given by [2]

$$[\mathrm{Im}(q_{11})]_{\mathrm{top}} = \mathrm{TTOP} = 2\left[\frac{1 \times 10^{-3} A_{\mathrm{loss}}}{20k \log_{10}(e)} + \frac{\eta}{2}\right]\left(\frac{k}{\alpha_1}\right)^{2/3} \ .$$

In this equation for determining the top of the search rectangle, A_{loss} is the maximum attenuation rate in dB/km for the modes to be found, $k = \frac{\omega}{c}$ is the free space wave number, η is the tropospheric absorptivity, and α_1 is the slope of

the duct in the bottom layer. The left and right edges of the initial search rectangle are located at

$$[\mathrm{Re}(q_{11})]_{\mathrm{left}} = \mathrm{TLEFT} = \mathrm{TLOLD} = 0.0$$

and

$$[\mathrm{Re}(q_{11})]_{\mathrm{right}} = \mathrm{TRIGHT} = \mathrm{TROLD} = 2.0 \ .$$

Because of the discontinuity in $\hat{k}_1(q_{11})$ and $\hat{k}_2(q_{11})$ along the imaginary q_{11}-axis introduced by our implementation of surface roughness, it is necessary for the initial search rectangle to have its left edge on the imaginary axis. For further details see the description of subroutine KHATX. The location of the right edge of the initial search rectangle can be changed by changing the value of TROLD.

The Shellman-Morfitt root-finding routine is used to find the zeros of the modal function in each search rectangle. In order to find the zeros along the real axis, an offset of 0.003 is introduced in subroutines FCTVLX and FDFDTX. This offset in the modal eigenvalues is corrected in MAIN by the statement

```
EIGEN(LK) = EIGEN(LK) - I* 0.003.
```

See the description of subroutine FCTVLX for more details.

After using the Shellman-Morfitt root-finding routine to search the initial search rectangle for modal eigenvalues, MAIN forms a new search rectangle to the left of the original by the statements

```
TLEFT = TLOLD - TLSTEP
TRIGHT = TLOLD
TLOLD = TLEFT
```

The width of the search rectangles formed to the left of the original can be varied by changing TLSTEP. If no zeros are found in NLEFT consecutive search rectangles, the search for modal eigenvalues to the left of the initial search rectangle is stopped. Next, MAIN starts forming search rectangles to the right of the initial search rectangle by using the statements

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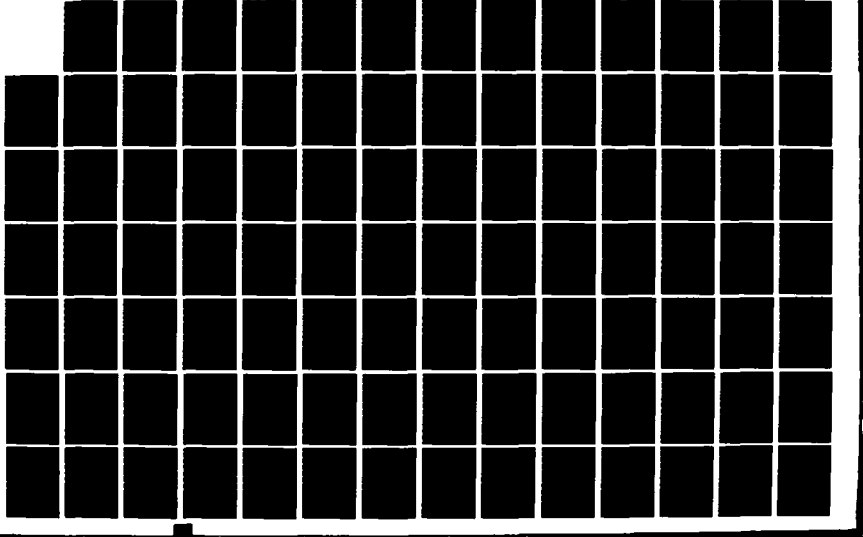
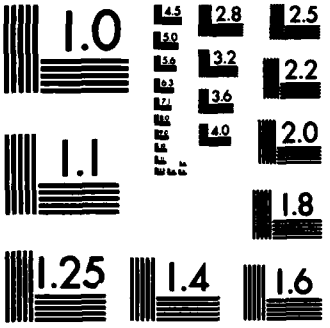




TLEFT = TROLD

TRIGHT = TROLD + TRSTEP

TROLD = TRIGHT.

The width of the search rectangles formed to the right of the original can be varied by changing TRSTEP. If no zeros are found in NRIGHT consecutive search rectangles, the search for modal eigenvalues to the right of the initial search rectangle is stopped.

The complete set of modes found by using the Shellman-Morfitt root-finding routine are stored in the array ZEROS. This array can hold up to NRZ elements. The value of NRZ can be changed by changing the PARAMETER statement in MAIN. If more than NRZ modes are found, MAIN will write a message indicating the search rectangles where the total number of modes exceeded NRZ. Only the first NRZ modes found are stored in ZEROS. After all the modes have been found, the modes stored in the array ZEROS are arranged by order of increasing real part, and any duplicate modes, i.e., modes within a distance TOL = 0.0001 of each other, are eliminated. These modes are then used by subroutine MODSUM to evaluate the field strength in dB relative to the free space value.

Calling Program Element: None.

Program Elements Called: CASIN, DHORIZ, FZEROX, MODSUM.

Common Block Areas: COMONE, COMTWO, COMTHR, COMFOR, COMFIV, COMSIX, COMSEV, COMEIG, COMNIN.

CADD

Description: Subroutine CADD adds two extended complex numbers of the form

$$z = \xi e^{\phi} ,$$

where ξ is a complex amplitude and ϕ a real exponent. If z_1 and z_2 are two extended complex numbers, their sum is

$$z = z_1 + z_2$$
$$= \xi e^{\phi} ,$$

where, assuming $\phi_1 \geq \phi_2$,

$$\xi = \xi_1 + \xi_2 \exp(\phi_2 - \phi_1)$$

and

$$\phi = \phi_1 .$$

The complex amplitude ξ is not normalized in any way.

Call Statement: Subroutine CADD is called by the statement

CALL CADD (ZA, ZE, XA, XE, YA, YE).

The input variables of subroutine CADD are:

XA - The complex amplitude of the extended complex number x.

XE - The real exponent of the extended complex number x.

YA - The complex amplitude of the extended complex number y.

YE - The real exponent of the extended complex number y.

The output variables of subroutine CADD are:

ZA - The complex amplitude of the extended complex number z = x + y.

ZE - The real exponent of the extended complex number $z = x + y$.

Calling Program Elements: HTGAIN, FDFDTX, FCTVLX, KHATX, MODSUM.

Program Elements Called: None.

Common Block Areas: None.

ADDX

Description: Subroutine ADDX adds two extended real numbers of the form

$$z = \xi e^{\phi} ,$$

where ξ is a real amplitude and ϕ a real exponent. If z_1 and z_2 are two extended real numbers, their sum is

$$\begin{aligned} z &= z_1 + z_2 \\ &= \xi e^{\phi} , \end{aligned}$$

where, assuming $\phi_1 \geq \phi_2$,

$$\xi = \xi_1 + \xi_2 \exp(\phi_2 - \phi_1)$$

and

$$\phi = \phi_1 .$$

The real amplitude ξ is not normalized in any way.

Call Statement: Subroutine ADDX is called by the statement

CALL ADDX (ZA, ZE, XA, XE, YA, YE).

The input variables of subroutine ADDX are:

XA - The real amplitude of the extended real number x.

XE - The real exponent of the extended real number x.

YA - The real amplitude of the extended real number y.

YE - The real exponent of the extended real number y.

The output variables of subroutine ADDX are:

ZA - The real amplitude of the extended real number z = x + y.

ZE - The real exponent of the extended real number z = x + y.

Calling Program Element: MODSUM.

Program Elements Called: None.

Common Block Areas: None.

HNKLX

Description: Subroutine HNKLX evaluates the modified Hankel functions of order one-third, $h_1(z)$ and $h_2(z)$, and their derivatives in extended complex arithmetic.

Call Statement: Subroutine HNKLX is called by the statement

CALL HNKLX (Z, H1A, H1E, H2A, H2E, DH1A, DH1E, DH2A, DH2E).

The input variable of subroutine HNKLX is:

Z - The location in the complex plane where the modified Hankel functions of order one-third and their derivatives are to be evaluated.

The output variables of subroutine HNKLX are:

H1A - The complex amplitude of $h_1(z)$ in extended complex arithmetic.
H1E - The real exponent of $h_1(z)$ in extended complex arithmetic.
H2A - The complex amplitude of $h_2(z)$.
H2E - The real exponent of $h_2(z)$.
DH1A - The complex amplitude of $h_1'(z)$.
DH1E - The real exponent of $h_1'(z)$.
DH2A - The complex amplitude of $h_2'(z)$.
DH2E - The real exponent of $h_2'(z)$.

Calling Program Elements: HTGAIN, FDFDTX, FCTVLX.

Program Elements Called: None.

Common Block Areas: None.

KHNKLX

Description: Subroutine KHNKLX evaluates the modified Hankel functions $k_1(z)$ and $k_2(z)$ and their derivatives in extended complex arithmetic.

Call Statement: Subroutine KHNKLX is called by the statement

CALL KHNKLX (Z, K1A, K1E, K2A, K2E, DK1A, DK1E, DK2A, DK2E).

The input variable of subroutine KHNKLX is:

Z - The location in the complex plane where the modified Hankel functions and their derivatives are to be evaluated.

The output variables of subroutine KHNKLX are:

K1A - The complex amplitude of $k_1(z)$ in extended complex arithmetic.
K1E - The real exponent of $k_1(z)$ in extended complex arithmetic.
K2A - The complex amplitude of $k_2(z)$.
K2E - The real exponent of $k_2(z)$.
DK1A - The complex amplitude of $k_1'(z)$.
DK1E - The real exponent of $k_1'(z)$.
DK2A - The complex amplitude of $k_2'(z)$.
DK2E - The real exponent of $k_2'(z)$.

Calling Program Elements: HTGAIN, FDFDTX, FCTVLX.

Program Elements Called: None.

Common Block Areas: None.

KHATX

Description: Subroutine KHATX evaluates the modified Hankel function combinations $\hat{k}_1(q_{11})$ and $\hat{k}_2(q_{11})$ and their derivatives in extended complex arithmetic. For the surface roughness model used in this program, the rms surface bump height δ was always taken to be zero when $Re(q_{11}) < 0$ since modes with $Re(q_{11}) < 0$ correspond to well-trapped modes and should be unaffected by the surface. This creates a discontinuity in $\hat{k}_1(q_{11})$ and $\hat{k}_2(q_{11})$ and hence, in the modal function, at $Re(q_{11}) = 0$ when $\delta \neq 0$. In order to use the Shellman-Morfitt root-finding routine with this discontinuity, when $Re(q_{11}) = 0$, subroutine KHATX has to know whether the current search rectangle lies to the left or right of the imaginary q_{11} axis and to evaluate $\hat{k}_1(q_{11})$ and $\hat{k}_2(q_{11})$ so that they will be continuous over the entire search rectangle. The information needed to do this is passed to KHATX from MAIN in the form of the flag INTCHK. When the initial search rectangle is set up, it is located at $0 \leq Re(q_{11}) \leq$ TROLD and INTCHK = 0. For search rectangles to the left of the initial search rectangle, that is for $Re(q_{11}) \leq 0$, INTCHK = 1. When $Re(q_{11}) = 0$, KHATX tests INTCHK to determine whether the search rectangle is to the left or right of the imaginary q_{11} axis and evaluates the proper "branch" of $\hat{k}_1(q_{11})$ and $\hat{k}_2(q_{11})$.

Call Statement: Subroutine KHATX is called by the statement

```
CALL KHATX (Q11, KHA1, KHE1, KHA2, KHE2, DKHA1, DKHE1, DKHA2, DKHE2).
```

The input variable of subroutine KHATX is:

Q11 - The location in the complex q_{11}-plane where $\hat{k}_1(q_{11})$ and $\hat{k}_2(q_{11})$ and their derivatives are to be evaluated.

The output variables of subroutine KHATX are:

KHA1 - The complex amplitude of $\hat{k}_1(q_{11})$ in extended complex arithmetic.

KHE1 - The real exponent of $\hat{k}_1(q_{11})$ in extended complex arithmetic.

KHA2 - The complex amplitude of $\hat{k}_2(q_{11})$.

KHE2 - The real exponent of $\hat{k}_2(q_{11})$.

DKHA1 - The complex amplitude of $\hat{k}_1'(q_{11})$.

DKHE1 - The real exponent of $\hat{k}_1'(q_{11})$.

DKHA2 - The complex amplitude of $\hat{k}_2'(q_{11})$.

DKHE2 - The real exponent of $\hat{k}_2'(q_{11})$.

Calling Program Elements: HTGAIN, FDFDTX, FCTVLX.

Program Elements Called: KHNKLX, CADD, CTANH, CSECH2.

Common Block Areas: COMTWO, COMTHR, COMFOR, COMFIV, COMSIX, COMSEV, COMEIG, COMNIN.

CTANH

Description: The complex function subroutine CTANH evaluates the complex hyperbolic tangent of the complex number z. If

$$|\mathrm{Re}(z)| > \frac{1}{10\sqrt{2}} \text{ or } |\mathrm{Im}(z)| > \frac{1}{10\sqrt{2}} ,$$

then tanh(z) is evaluated by using

$$\tanh(z) = \frac{e^{z} - e^{-z}}{e^{z} + e^{-z}} .$$

If

$$|\mathrm{Re}(z)| \leq \frac{1}{10\sqrt{2}} \text{ and } |\mathrm{Im}(z)| \leq \frac{1}{10\sqrt{2}} ,$$

then tanh(z) is evaluated by using the first five terms of the Taylor series expansion

$$\tanh(z) = z - \frac{1}{3}z^{3} + \frac{2}{15}z^{5} - \frac{17}{315}z^{7} + \frac{62}{2835}z^{9} - \cdots .$$

Calling Statement: The complex function subroutine CTANH is called by

CTANH(Z).

The input variable of function subroutine CTANH is:

Z - A complex number.

The output variable of function subroutine CTANH is:

CTANH - The hyperbolic tangent of the complex number z.

Calling Program Element: KHATX.

Program Elements Called: None.

Common Block Areas: None.

CSECH2

Description: The complex function subroutine CSECH2 evaluates the square of the hyperbolic secant of the complex variable z by using

$$\operatorname{sech}^2(z) = \frac{4}{e^{2z} + 2 + e^{-2z}} .$$

Calling Statement: The complex function subroutine CSECH2 is called by

CSECH2(Z).

The input variable of function subroutine CSECH2 is:

Z - A complex number.

The output variable of function subroutine CSECH2 is:

CSECH2 - The square of the hyperbolic secant of the complex variable z.

Calling Program Element: KHATX.

Program Elements Called: None.

Common Block Areas: None.

FCTVLX

Description: Subroutine FCTVLX evaluates the modal function $||\underset{\sim}{\Delta}(q_{11})||$ in extended complex arithmetic. When using the Shellman-Morfitt root-finding routine to locate the zeros of a complex function, it is assumed that these zeros do not fall on a mesh line. Since the imaginary part of some of the zeros of $||\underset{\sim}{\Delta}(q_{11})||$ are numerically indistinguishable from zero, the real q_{11} axis should not be a mesh line. However, because of the way the Shellman-Morfitt root-finding routine sets up mesh squares, if the real axis passes through a search rectangle, the real axis will always be a mesh line. One way to avoid problems caused by this is to find the zeros of the functions $||\underset{\sim}{\Delta}(t_{11})||$ where

$$t_{11} = q_{11} - i\Delta q \ . \tag{8.1}$$

The shift Δq is chosen to satisfy

$$\delta_T < \Delta q < \frac{1}{2n_{max}}\delta_m \ , \tag{8.2}$$

where δ_T is the tolerance to which zeros are located, δ_m is the initial mesh size, and n_{max} is the maximum number of times that subroutine FZEROX will reduce the mesh size by a factor of two. In XWVG where $\delta_T = 0.0001$, $\delta_m = 0.04$, and $n_{max} = 2$, a value $\Delta q = 0.003$ is chosen. With Δq chosen to satisfy (8.2), even if subroutine FZEROX reduces the mesh size by a factor of two n_{max} times, the values of t_{11} corresponding to $Im(q_{11}) = 0$ will still not lie on a mesh line. From equation (8.1) it is seen that the zeros of $||\underset{\sim}{\Delta}(q_{11})||$ that were along the line $Im(q_{11}) = 0$ are now shifted to the line $Im(q_{11}) = + i\Delta q$. Thus, the zeros of $||\underset{\sim}{\Delta}(t_{11})||$ are related to the zeros of $||\underset{\sim}{\Delta}(q_{11})||$ by

$$q_{11} = t_{11} - i\Delta q \ . \tag{8.3}$$

The shift to the new variable t_{11} occurs inside subroutine FCTVLX as well as inside subroutine FDFDTX. The shift back to q_{11}, equation (8.3), occurs in MAIN. Subroutine EXCFAC, which uses subroutine FDFDTX, also has a statement that compensates for the shift, Δq.

Calling Statement: Subroutine FCTVLX is called by the statement

CALL FCTVLX (Q11, DETA, DETE).

The input variable of subroutine FCTVLX is:

Q11 - The location in the complex q_{11}-plane where the modal function is to be evaluated.

The output variables of subroutine FCTVLX are:

DETA - The complex amplitude of the modal function in extended complex arithmetic.

DETE - The real exponent of the modal function in extended complex arithmetic.

Calling Program Element: FINDFX.

Program Elements Called: HNKLX, KHNKLX, KHATX, CADD.

Common Block Areas: COMONE, COMTWO.

FDFDTX

Description: Subroutine FDFDTX evaluates the modal function $||\underset{\sim}{\Delta}(q_{11})||$ and its derivative

$$\frac{\partial}{\partial q_{11}}\{||\underset{\sim}{\Delta}(q_{11})||\}$$

in extended complex arithmetic. As in subroutine FCTVLX, the value of q_{11} is shifted inside the subroutine to

$$t_{11} = q_{11} - i\Delta q \ ,$$

where $\Delta q = 0.003$. This shift by Δq is compensated for in MAIN when FDFDTX is used by the Shellman-Morfitt root-finding routine. When FDFDTX is used in the evaluation of the modal excitation factors, this shift by Δq is compensated for in subroutine EXCFAC.

Calling Statement: Subroutine FDFDTX is called by the statement

CALL FDFDTX (Q11, DETA, DETE, DETPA, DETPE).

The input variable of subroutine FDFDTX is:

Q11 - The location in the complex q_{11}-plane where the modal function and its derivative are to be evaluated.

The output variables of subroutine FDFDTX are:

DETA - The complex amplitude of the modal function in extended complex arithmetic.

DETE - The real exponent of the modal function in extended complex arithmetic.

DETPA - The complex amplitude of the derivative of the modal function.

DETPE - The real exponent of the derivative of the modal function.

Calling Program Elements: NOMSHX, EXCFAC.

Program Elements Called: HNKLX, KHNKLX, KHATX, CADD.

Common Block Areas: COMONE, COMTWO.

FZEROX

Description: Subroutine FZEROX is the main subroutine of the Shellman-Morfitt root-finding routine modified for extended complex arithmetic and the trilinear tropospheric duct model. The other subroutines comprising the Shellman-Morfitt root-finding routine are subroutines QUAD, NOMSHX, and FINDFX. Subroutine FZEROX divides a specified rectangular region of the complex plane into mesh squares and determines approximate values for the complex zeros of an analytic function f(z). These approximate values for the zeros of f(z) are improved by subroutine NOMSHX by using Newton-Raphson iteration. The function f(z) is assumed to have no poles or branch cuts in the rectangular region of the complex plane being searched. The zeros of f(z) in the search region are assumed to be simple zeros. Other assumptions made are that the zeros of f(z) do not fall on the mesh lines and that the lines Im[f(z)] = 0 and Re[f(z)] = 0 do not coincide with mesh lines. The maximum number of crossings of Im[f(z)] = 0 with any one edge of the search rectangle that is currently allowed by FZEROX is 50. This value can be changed by changing the dimensions of KEDGE1, KEDGE2, KEDGE3, and KEDGE4.

Because of the discontinuity of the modal function along the imaginary axis that is introduced by the discontinuity in the surface roughness formula, special care has to be taken when finding the complex roots of the modal function in a search rectangle that has TLEFT = 0.0 or TRIGHT = 0.0. Since FZEROX will normally extend the search rectangle by one mesh square in all directions, this expansion to the left or right has to be suppressed when TLEFT = 0.0 or TRIGHT = 0.0. This is done with the statements

```
IF(TLEFT.EQ.0.0)    JLT = 0
```

and

```
IF(TRIGHT.EQ.0.0)   JRT = 0
```

in subroutine FZEROX. When the modal function $||\underset{\sim}{\Delta}(q_{11})||$ is evaluated along the imaginary axis, the value of q_{11} is offset by an amount SHIFT = $\pm$ 1.0 $\times$ 10^{-9} in subroutine FINDFX. The plus sign is taken if the search rectangle

lies to the right of the imaginary axis and the minus sign is taken if the search rectangle lies to the left of the imaginary axis.

Calling Statement: Subroutine FZEROX is called by the statement

CALL FZEROX (TLEFT, TRIGHT, TBOT, TTOP, TMSH0, TOL, MPRINT, ZEROS, NRZ).

The input variables of subroutine FZEROX are:

TLEFT - The value of the real part of z at the left edge of the search rectangle.

TRIGHT - The value of the real part of z at the right edge of the search rectangle.

TBOT - The value of the imaginary part of z at the bottom edge of the search rectangle.

TTOP - The value of the imaginary part of z at the top edge of the search rectangle.

TMSH0 - The initial size of the mesh squares used to divide the search rectangle. TMSH0 should be small enough so that the variation of Im[f(z)] and Re[f(z)] along the mesh square edges is approximately linear. Subroutine FZEROX assumes that this is true. The smaller TMSH0 is, the longer the program running time. Under some circumstances FZEROX will automatically reduce the mesh size by a factor of 2 a total of MAXNT = 2 times. If FZEROX still experiences difficulties after these reductions in mesh size, an error message will be output and the program will stop. FZEROX allows a maximum number, MAXNSQ, of mesh squares to lie on any one phase curve Im[f(z)] = 0 in the search rectangle. If this number is exceeded, an error message is output and the program stops. MAXNSQ is currently set at four times the number of mesh squares along the longest side of the search rectangle.

TOL - The tolerance to which zeros are located. Zeros located closer together than TOL cannot be distinguished.

MPRINT - A flag for debugging output. If MPRINT = 0, no debugging printout is given; while if MPRINT = 1, debugging printout is given. A value of MPRINT = 0 is recommended except when absolutely necessary, because of the large amount of printout produced.

The output variables of subroutine FZEROX are:

ZEROS - An array containing the complex zeros of f(z) in the specified rectangular region of the complex plane. The array ZEROS assumes the dimensions specified in MAIN.

NRZ - The number of complex zeros of f(z) in the specified rectangular region of the complex plane.

Calling Program Element: MAIN.

Program Elements Called: QUAD, NOMSHX, FINDFX.

Common Block Areas: NEWMSH, TMCCOM.

QUAD

Description: Subroutine QUAD is an auxiliary subroutine in the Shellman-Morfitt root-finding routine. The other subroutines comprising the Shellman-Morfitt root-finding routine are FZEROX, NOMSHX, and FINDFX. Subroutine QUAD finds the real roots of a quadratic equation of the form

$$ax^2 + 2bx + c = 0.$$

Calling Statement: Subroutine QUAD is called by the statement

```
CALL QUAD (A, B, C, SOL, NRSOL).
```

The input variables of subroutine QUAD are:

A - The coefficient of x^2 in the quadratic equation.

B - The coefficient of 2x in the quadratic equation.

C - The constant term in the quadratic equation.

The output variables of subroutine QUAD are:

SOL - An array containing the real roots of the quadratic equation.

NRSOL - The number of real roots of the quadratic equation found. A value of NRSOL = 1 results when the quadratic equation approaches linearity.

Calling Program Element: FZEROX.

Program Elements Called: None.

Common Block Areas: None.

NOMSHX

Description: Subroutine NOMSHX is an auxiliary subroutine of the Shellman-Morfitt root-finding routine modified for extended complex arithmetic. The other subroutines that comprise the Shellman-Morfitt root-finding routine are FZEROX, FINDFX, and QUAD. Subroutine NOMSHX takes the approximate locations of the complex zeros of f(z) provided by subroutine FZEROX and improves them by using Newton-Raphson iteration.

Calling Statement: Subroutine NOMSHX is called by the statement

CALL NOMSHX (TMESH, TOL, MPRINT, ZEROS, NRZ).

The input variables of subroutine NOMSHX are:

TMESH - The size of the mesh squares used to divide the search rectangle.

TOL - The tolerance to which zeros are to be located. Zeros located closer than TOL cannot be distinguished.

MPRINT - A flag for debugging output. If MPRINT = 0, no debugging printout is given; while if MPRINT = 1, debugging printout is given.

ZEROS - An array containing the approximate location of the complex zeros of f(z).

NRZ - The number of complex zeros stored in array ZEROS.

The output variable of subroutine NOMSHX is:

ZEROS - An array containing the Newton-Raphson iterated values for the complex zeros of f(z). ZEROS is used for both input and output.

Calling Program Element: FZEROX.

Program Elements Called: FDFDTX.

Common Block Areas: NEWMSH.

FINDFX

Description: Subroutine FINDFX is an auxiliary subroutine of the Shellman-Morfitt root-finding routine modified for extended complex arithmetic and the trilinear tropospheric duct propagation model. The other subroutines that comprise the Shellman-Morfitt root-finding routine are FZEROX, QUAD, and NOMSHX. Subroutine FINDFX takes the coordinates of the mesh square corners and evaluates the modal function $||\underset{\sim}{\Delta}(q_{11})||$ there. The modal function is evaluated in extended complex arithmetic. Because of the discontinuity of the modal function along the imaginary q_{11} axis introduced by the surface roughness formulation, special care has to be taken when finding the complex zeros of $||\underset{\sim}{\Delta}(q_{11})||$ in a search rectangle that has TLEFT = 0.0 or TRIGHT = 0.0. The Shellman-Morfitt root-finding routine requires that $||\underset{\sim}{\Delta}(q_{11})||$ have no poles, branch cuts, or other discontinuities in the search rectangle. In order to ensure that this condition is met when the search rectangle has TLEFT = 0.0 or TRIGHT = 0.0, a small offset value OFFSET = 1.0×10^{-9} is introduced. If TLEFT = 0.0, then instead of evaluating $||\underset{\sim}{\Delta}(q_{11})||$ along $\mathrm{Re}(q_{11}) = 0$, $||\underset{\sim}{\Delta}(q_{11})||$ is evaluated along $\mathrm{Re}(q_{11})$ = + OFFSET. If TRIGHT = 0.0, then instead of evaluating $||\underset{\sim}{\Delta}(q_{11})||$ along $\mathrm{Re}(q_{11}) = 0$, $||\underset{\sim}{\Delta}(q_{11})||$ is evaluated at $\mathrm{Re}(q_{11})$ = - OFFSET.

Calling Statement: Subroutine FINDFX is called by the statement

```
CALL FINDFX (JR, JI, F, FE, TLEFT, TRIGHT).
```

The input variables of subroutine FINDFX are:

JR - The real part of the mesh square corner coordinates in mesh units.

JI - The imaginary part of the mesh square corner coordinates in mesh units.

TLEFT - The value of the real part of q_{11} at the left edge of the search rectangle.

TRIGHT - The value of the real part of q_{11} at the right edge of the search rectangle.

The output variables of subroutine FINDFX are:

F - The complex amplitude of the modal function in extended complex arithmetic.

FE - The real exponent of the modal function in extended complex arithmetic.

Calling Program Element: FZEROX.

Program Elements Called: FCTVLX.

Common Block Areas: TMCCOM.

EXCFAC

Description: Subroutine EXCFAC calculates the modal excitation factor $\lambda_n(q_{11})$ given by equation (2.104) in extended complex arithmetic. Because of the shift in q_{11} by the amount $-i\Delta q = -0.003i$ built into subroutine FDFDTX, before EXCFAC calls FDFDTX it shifts q_{11} by the amount $+i\Delta q = +0.003i$. This exactly compensates for the shift inside FDFDTX.

Calling Statement: Subroutine EXCFAC is called by the statement

CALL EXCFAC (XFAC, XFACE, Q11).

The input variable of subroutine EXCFAC is:

Q11 - The modal eigenvalue for which the excitation factor is to be evaluated.

The output variables of subroutine EXCFAC are:

XFAC - The complex amplitude of the excitation factor in extended complex arithmetic.

XFACE - The real exponent of the excitation factor in extended complex arithmetic.

Calling Program Element: MODSUM.

Program Elements Called: FDFDTX.

Common Block Areas: COMTWO, COMTHR.

HTGAIN

Description: Subroutine HTGAIN calculates the modal height-gain functions $E(n, z_R, z_T)$ corresponding to the n-th modal eigenvalue q_{11} for the trilinear tropospheric duct propagation model. The height-gain functions are given by equations (2.150-2.166).

Calling Statement: Subroutine HTGAIN is called by the statement

CALL HTGAIN (ENZA, ENZE, Q11, ZRCVR, ZXMTR).

The input variables of subroutine HTGAIN are:

Q11 - The modal eigenvalue for which the height-gain function is to be calculated.

ZRCVR - The height in meters of the receiver above ground level.

ZXMTR - The height in meters of the transmitter above ground level.

The output variables of subroutine HTGAIN are:

ENZA - The complex amplitude of the height-gain function in extended complex arithmetic.

ENZE - The real exponent of the height-gain function in extended complex arithmetic.

Calling Program Element: MODSUM.

Program Elements Called: HNKLX, KHNKLX, KHATX, CADD.

Common Block Areas: COMONE, COMTWO, COMTHR.

MODSUM

Description: Subroutine MODSUM calculates the coherent and incoherent mode sum field strength values in dB relative to the free space value. These field strength values are calculated by using equations (2.108) and (2.109).

Calling Statement: Subroutine MODSUM is called by the statement

CALL MODSUM (ECMS, EIMS, RNG, ZRCVR, ZXMTR, ZEROS, NRZ, NRMODE).

The input variables of subroutine MODSUM are:

RNG - The range in kilometers at which the field strength is to be evaluated.

ZRCVR - The height in meters of the receiver.

ZXMTR - The height in meters of the transmitter.

ZEROS - An array containing the complex modal eigenvalues q_{11}.

NRZ - The dimension of the complex array ZEROS.

NRMODE - The number of modal eigenvalues contained in array ZEROS.

The output variables of subroutine MODSUM are:

ECMS - The coherent mode sum field strength value in dB relative to the free space value.

EIMS - The incoherent mode sum field strength value in dB relative to the free space value.

Calling Program Element: MAIN.

Program Elements Called: EXCFAC, HTGAIN, CADD, ADDX.

Common Block Areas: COMTWO, COMTHR.

SEAH20

Description: Subroutine SEAH20 evaluates the effective relative dielectric constant ε_r and the effective conductivity σ of seawater as a function of temperature, salinity, and frequency. The complex refractive index of seawater is then given by

$$n_g^2 = \varepsilon_r - i \frac{\sigma}{2\pi f \varepsilon_o} ,$$

where f is the frequency and ε_o is the permittivity of free space. The model used to evaluate ε_r and σ is from the paper by Klein and Swift [24]. There is a discontinuity in the values of ε_r and σ calculated by this model as the salinity goes to zero.

Calling Statement: Subroutine SEAH20 is called by the statement

CALL SEAH20 (SIGEFF, EPSEFF, T, S, FREQ).

The input variables of subroutine SEAH20 are:

T - The temperature of the seawater in degrees Celsius. The temperature is assumed to be in the interval -2°C to +40°C.

S - The salinity of the seawater in grams of salt per kilogram of seawater. The salinity is assumed to be in the interval 0 to 40 g salt/kg seawater.

FREQ - The frequency in hertz. The frequency is assumed to be in the range 0 to 300 GHz.

The output variables of subroutine SEAH20 are:

SIGEFF - The effective conductivity of the seawater in siemens per meter.

EPSEFF - The effective relative dielectric constant.

Calling Program Element: MAIN.

Program Elements Called: None.

Common Block Areas: None.

DHORIZ

Description: Subroutine DHORIZ calculates the radio horizon distance for a receiver at height Z_R and a transmitter at height Z_T. The horizon distance is given by

$$d = \sqrt{2r_{eff}z_R} + \sqrt{2r_{eff}z_T}$$

The effective earth radius r_{eff} is related to the true earth radius r_t by

$$r_{eff} = \frac{4}{3}r_t .$$

Calling Statement: Subroutine DHORIZ is called by the statement

CALL DHORIZ (DHZ, ZRCVR, ZXMTR).

The input variables of subroutine DHORIZ are:

ZRCVR - The receiver height in meters.

ZXMTR - The transmitter height in meters.

The output variable of subroutine DHORIZ is:

DHZ - The radio horizon distance in kilometers for the given transmitter and receiver heights.

Calling Program Element: MAIN.

Program Elements Called: None.

Common Block Areas: None.

CASIN

Description: The complex function subroutine CASIN evaluates the complex arcsine of a complex number z by using the equation

$$\sin^{-1}(z) = -i \ln [(1 - z^2)^{1/2} + iz] .$$

Calling Statement: The complex function subroutine CASIN is called by

CASIN(Z).

The input variable of function subroutine CASIN is:

Z - A complex number.

The output variable of function subroutine CASIN is:

CASIN - The complex arcsine of the complex number z.

Calling Program Element: MAIN.

Program Elements Called: None.

Common Block Areas: None.

9. SOURCE LISTING FOR XWVG

```
1 C**********
2 C      THIS PROGRAM FINDS THE COMPLEX MODES IN THE C11-PLANE FOR A
3 C      TRILINEAR MODEL OF THE REFRACTIVITY PROFILE OF THE TROPHOSPHERE.
4 C      THESE MODES ARE FOUND USING THE COMPLEX ROOT FINDING ROUTINE OF
5 C      MORFITT AND SHELLMAN.
6 C      THE MODES IN THE COMPLEX C11-PLANE ARE CONVERTED TO
7 C      VALUES IN THE COMPLEX THETA-PLANE.
8 C      BOTH THE COHERENT AND INCOHERENT MODE SUM VALUES OF THE
9 C      FIELD STRENGTH RELATIVE TO FREE SPACE ARE EVALUATED
10 C
11 C       INPUTS...
12 C          MFILE - IF MFILE=0 CALCULATE EIGENVALUES
13 C                  IF MFILE=1 READ EIGENVALUES FROM LOGICAL FILE 12
14 C
15 C          MPOL - IF MPOL=0 HORIZONTAL POLARIZATION
16 C                 IF MPOL=1 VERTICAL POLARIZATION
17 C          FREQ - FREQUENCY (IN MHZ) FOR WHICH MODES ARE TO BE FOUND
18 C          ALPHA1 - SLOPE (IN UNITS OF 1/METER) OF FIRST(LOWEST) LAYER
19 C            IN TRILINEAR MODEL. MUST BE POSITIVE.
20 C          ALPHA2 - SLOPE OF SECOND(MIDDLE) LAYER IN TRILINEAR MODEL. MAY
21 C            BE POSITIVE OR NEGATIVE.
22 C          ALPHA3 - SLOPE OF THIRD(TOP) LAYER IN TRILINEAR MODEL. MUST BE
23 C            POSITIVE
24 C          Z2 - HEIGHT (IN METERS) OF INTERFACE BETWEEN FIRST AND SECOND
25 C            LAYERS OF TRILINEAR MODEL. MUST BE POSITIVE.
26 C          Z3 - HEIGHT (IN METERS) OF INTERFACE BETWEEN SECOND AND THIRD
27 C            LAYERS OF TRILINEAR MODEL.  MUST BE GREATER THAN Z2.
28 C            TO FIELD STRENGTH LEVEL FOR WHICH MODES WILL BE FOUND.
29 C          ZREF - HEIGHT (IN METERS) AT WHICH THE VALUE OF THE
30 C            MODIFIED REFRACTIVITY, M(Z), IS SPECIFIED
31 C            ZREF IS ASSUMED TO BE IN THE FIRST LAYER, I.E.
32 C                          (0.0).LE.(ZREF).LE.(Z2)
33 C          REFM - VALUE OF THE MODIFIED REFRACTIVITY (IN M-UNITS)
34 C            AT THE REFERENCE HEIGHT ZREF
35 C
36 C          ETA - ABSORPTIVITY OF TROPOSPHERE. ZERO OR NEGATIVE.
37 C          TEMP - SEAWATER TEMPERATURE (IN DEGREES CELSIUS)
38 C          SALT - SEAWATER SALINITY (IN GRAMS SALT / KG SEAWATER)
39 C          DELTA - R.M.S. BUMP HEIGHT (IN METERS) OF THE GROUND
40 C          ALOSS - MAXIMUM LOSS RATE (IN DB/KM) OF MODES WHICH WILL BE
41 C            FOUND.
42 C
43 C          RNGINT - INITIAL RANGE (IN KILOMETERS) AT WHICH FIELD STRENGTH
44 C            IS TO BE CALCULATED.
45 C          NRNG - NUMBER OF RANGE VALUES AT WHICH FIELD STRENGTH IS TO BE
46 C            CALCULATED.
47 C          RNGSTP - INCREMENT (IN KILOMETERS) BY WHICH THE RANGE CHANGES.
48 C
49 C          ZRINIT - HEIGHT(IN METERS) AT WHICH THE RECEIVER IS
50 C            INITIALLY LOCATED.
51 C          NZRCVR - NUMBER OF RECEIVER HEIGHTS AT WHICH THE
52 C            FIELD STRENGTH IS TO BE CALCULATED.
53 C          ZRSTEP - INCREMENT (IN METERS) BY WHICH THE RECEIVER
54 C            HEIGHT CHANGES.
55 C
```

```
C         ZXINIT - HEIGHT (IN METERS) AT WHICH THE TRANSMITTER
C           IS INITIALLY LOCATED
C         NZXMTR - NUMBER OF TRANSMITTER HEIGHTS AT WHICH THE
C           FIELD STRENGTH IS TO BE CALCULATED.
C         ZXSTEP - INCREMENT (IN METERS) BY WHICH THE TRANSMITTER
C           HEIGHT CHANGES
C
C     OUTPUTS...
C         ZEROS - ARRAY CONTAINING MODES IN THE COMPLEX SIN-PLANE FOR
C           THE GIVEN TRILINEAR PROFILE.
C         THETA - ARRAY CONTAINING CORRESPONDING MODES IN THE COMPLEX
C           THETA-PLANE.
C         ATNU - ATTENUATION (IN DB/KM) OF THE GIVEN MODE
C         FCMS - COHERENT MODE SUM FIELD STRENGTH  RELATIVE TO THE
C           FREE SPACE VALUE (IN DECIBELS).
C         FIMS - INCOHERENT MODE SUM FIELD STRENGTH RELATIVE TO THE
C           FREE SPACE VALUE (IN DECIBELS).
C
C     PARAMETER INPUTS...
C         NRZ - MAXIMUM NUMBER OF MODES EXPECTED TO BE FOUND
C
C     SUBROUTINES USED...
C         FZEROX,FINDEX,GUAD,NORMSX,RANKLX,KRNKLX,CASIN,FCTVLX,
C         FDEITX,CADD,ADDX,EXCFAC,HTGAIN,MODSUM,KRATX,CTANH,CSECH2,
C         DHORIZ
C**********
      IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
      DOUBLE PRECISION*6 KDEL2,KDEL6,KDEL10
      COMPLEX  ZEROS,THETA,SQNG,I,CASIN,EIGEN,CTEMP,RAT,CSQRT,BETASQ,
     *   TAU,TAUSR,TAU2,TAU3,TAU4,A0,A2,A4,A6,A8,A10,B1,B3,B5,B7,
     *   B9,B11,DA0,DA2,DA4,DA6,DA8,D0,D1,D3,D5,D7,D9,CNE,RNONG
      EXTERNAL CASIN
C****
C     PARAMETERS USED IN PROGRAM
C
      PARAMETER (NRZ=600)
C****
      DIMENSION  ZEROS(NRZ),THETA(NRZ),EIGEN(NRZ)
      COMMON/COMONE/C1,C3,C4,C5,C6,C7,C8,C9,SQNA2
      COMMON/COMTWO/ ETAABS,ALPHA1,K,C11,SQNG
      COMMON/COMTHR/Z2,Z3,A2MAG,ALPHA3,WAVENO,ALPHA2
      COMMON/COMFOR/DELTA,KDEL2,RNONG
      COMMON/COMFIV/A0,A2,A3,A4,A6,A7,A8,A10,A11
      COMMON/COMSIX/B0,B1,B3,B5,B7,B9,B11
       COMMON/COMSEV/DA0,DA1,DA2,DA4,DA5,DA6,DA8,DA9
       COMMON/COMEIG/D0,D1,D3,D5,D7,D9
       COMMON/COMMIN/INTCHK
       DATA  EPS0,C/8.85434E-12,2.9979250E8/
       DATA  I,PI,C20LOG/(0.0,1.0),3.14159265,8.68588964/
       DATA  CNE/(1.0,0.0)/
C*****
C     FUNCTION DEFINITION
C
       CBRT(Y) = Y**(1.0/3.0)
C
```

```
C****
  902 FORMAT(/6X,'IN THE SEARCH RECTANGLE'//10X,'TLEFT = ',F7.2,2X,
     * 'TRIGHT = ',F7.2/6X,'THE TOTAL NUMBER OF ZEROS NOW EXCEEDS '
     * 'NRZ BY ',I4/6X,'EXCESS ZEROS IN CURRENT SEARCH RECTANGLE '
     * 'WILL BE DISCARDED'/)
  903 FORMAT(////10X,'SOLUTIONS OF THE MODAL EQUATION'///)
  904 FORMAT(2X,'Q11(',I3,') = ',1P2D14.4,5X,'THETA(',I3,') = ',
     * 1P2E14.4,3X,'ATNU = ',1PD12.4,' DB/KM')
C
  800 FORMAT(////10X,'ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED',
     *   ' BY'/10X,'TLEFT = ',F7.2,5X,'TRIGHT = ',F7.2,5X,'TBOT = ',F7.2
     *   ,5X,'TTOP = ',F7.2//)
  801 FORMAT(20X,'EIGEN(',I3,') = ',1P2E14.4)
C
  905 FORMAT(1H1///5X,'FIELD STRENGTH RELATIVE TO FREE SPACE VALUE'///)
  907 FORMAT(5X,'RANGE',8X,'XMTR',8X,'RCVR',6X,'COHERENT',4X,
     *   'INCOHERENT',5X,'HORIZON'/5X,'(KM)',10X,'(M)',10X,'(M)',7X,
     *   'MODE SUM',5X,'MODE SUM',5X,'DISTANCE'/44X,'(DB)',9X,'(DB)',
     *   9X,'(KM)'///)
  908 FORMAT(2X,F10.2,3X,F10.3,3X,F10.3,3X,F10.4,3X,F10.4,3X,F10.2)
  930 FORMAT(//5X,'FREQ = ',F11.4,1X,'MHZ'/5X,'ALPHA1 = ',D14.4,' /M'/
     *   5X,'ALPHA2 = ',D14.4,' /M'/5X,'ALPHA3 = ',D14.4,' /M'/
     *   5X,'Z2 = ',F12.4,' M'/5X,'Z3 = ',F12.4,' M'/
     *   5X,'ALOSS = ',F12.4,' DB/KM'/)
  931 FORMAT(5X,'DELTA = ',1PD14.4,' M'/5X,'EPSGND = ',1PD14.4,
     *   ' FARADS/M'/5X,'SGMGND = ',1PD14.4,' MHOS/M'/5X,'ETA = ',
     *   1PD14.4///)
  932 FORMAT(5X,'SEAWATER TEMPERATURE = ',1PD14.4,' DEGREES CELSIUS'/
     *   5X,'SEAWATER SALINITY = ',1PD14.4,' GRAMS SALT/KG SEAWATER')
  970 FORMAT(1X,'Q11(',I4,')=',2D22.11)
  971 FORMAT(8X,I4)
  972 FORMAT(1X,'NRMODE=',I4)
  973 FORMAT(11X,2D22.11)
  976 FORMAT(5X,'HORIZONTAL POLARIZATION'/)
  977 FORMAT(5X,'VERTICAL POLARIZATION'/)
  979 FORMAT(5X,'M(',F6.2,') = ',F6.2/)
C****
C     READ INPUT VALUES FROM LOGICAL FILE NUMBER 13
C
      READ(13,) MFILE
      READ(13,) MPOL
      READ(13,) FREQ
      READ(13,) ALPHA1
      READ(13,) ALPHA2
      READ(13,) ALPHA3
      READ(13,) Z2
      READ(13,) Z3
      READ(13,) ZREF
      READ(13,) REFM
      READ(13,) ETA
      READ(13,) TEMP
      READ(13,) SALT
      READ(13,) DELTA
      READ(13,) ALOSS
      READ(13,) RNGINT
```

```
      READ(13,*) NRNG
      READ(13,*) RNGSTP
      READ(13,*) ZRINIT
      READ(13,*) NZRCVR
      READ(13,*) ZRSTEP
      READ(13,*) ZXINIT
      READ(13,*) NZXMTR
      READ(13,*) ZXSTEP
C****
C     WRITE DATA TO LOGICAL FILE NUMBER 11
C
      WRITE(11,930) FREQ,ALPHA1,ALPHA2,ALPHA3,Z2,Z3,ALOSS
      WRITE(11,979) ZREF,REFM
      IF(MPOL.EQ.0) THEN
      WRITE(11,976)
      ELSE
      WRITE(11,977)
      END IF
      FREQ=FREQ*1.0E6
      CALL SEAH2O(SGMGND,EPSREL,TEMP,SALT,FREQ)
      EPSGND=EPSREL*EPS0
      WRITE(11,932) TEMP,SALT
      WRITE(11,931) DELTA,EPSGND,SGMGND,ETA
C****
C     CALCULATE 'CONSTANTS' NEEDED FOR THE EVALUATION OF MODAL EQUATIONS
C
      ETAABS=ETA
      WAVENO  = 2.0*PI*FREQ/C
      A2MAG = ABS(ALPHA2)
      SGNA2=-SIGN(1.0,ALPHA2)
      C1 = CBRT(A2MAG/ALPHA1)
      C3 = CBRT(A2MAG/ALPHA3)
      C11 = CBRT(ALPHA1/WAVENO)
      C4 = ALPHA1*Z2/(C11*C11)
      C5 = CBRT(ALPHA1*ALPHA1/(A2MAG*A2MAG))
      TEMP = CBRT(WAVENO*WAVENO/(A2MAG*A2MAG))
      C6 = TEMP*ALPHA1*Z2
      C7 = TEMP*(ALPHA1*Z2+ALPHA2*(Z3-Z2))
      C8 = CBRT(ALPHA1*ALPHA1/(ALPHA3*ALPHA3))
      C9=(ALPHA1*Z2+ALPHA2*(Z3-Z2))*CBRT(WAVENO*WAVENO/(ALPHA3*ALPHA3))
      SQNG=EPSGND/EPS0-I*SGMGND/(EPS0*2.0*PI*FREQ)
      H=-2.0E-6*REFM/ALPHA1+ZREF
      C11SQ=C11*C11
C
C     THE FOLLOWING VALUES ARE NEEDED BY SUBROUTINE KHATX
C     THESE VALUES DEPEND UPON THE POLARIZATION OF THE FIELD
C
C     FOR HORIZONTAL POLARIZATION WE HAVE
C
      IF(MPOL.EQ.0) THEN
      RNONG=ONE
C
C     FOR VERTICAL POLARIZATION WE HAVE
C
      ELSE
```

```
      RNONG=(1.0-I*ETAABS-ALPHA1*H)/SQNG
      END IF
C
      TAU=SQNG-1.0-I*ETAABS+ALPHA1*H
      TAUSR=CSQRT(TAU)
      TAU2=TAU*TAU
      TAU3=TAU2*TAU
      TAU4=TAU2*TAU2
      KDEL2=WAVENO*WAVENO*DELTA*DELTA
      KDEL6=KDEL2**3
      KDEL10=KDEL2**5
      A0=RNONG*TAUSR
      A2=RNONG/(2.0*TAUSR)
      A3=KDEL2
      A4=-RNONG/(8.0*TAU*TAUSR)
      A6=RNONG/(16.0*TAU2*TAUSR)
      A7=-KDEL6/3.0
      A8=-5.0*RNONG/(128.0*TAU3*TAUSR)
      A10=7.0*RNONG/(256.0*TAU4*TAUSR)
      A11=2.0*KDEL10/15.0
      B0=1.0
      B1=RNONG*TAUSR*KDEL2
      B3=RNONG*KDEL2/(2.0*TAUSR)
      B5=RNONG*(-TAUSR*KDEL6/3.0-KDEL2/(8.0*TAU*TAUSR))
      B7=RNONG*(-KDEL6/(6.0*TAUSR)+KDEL2/(16.0*TAU2*TAUSR))
      B9=RNONG*(2.0*TAUSR*KDEL10/15.0+KDEL6/(24.0*TAU*TAUSR)
     $   -5.0*KDEL2/(128.0*TAU3*TAUSR))
      B11=RNONG*(KDEL10/(15.0*TAUSR)-KDEL6/(48.0*TAU2*TAUSR)
     $   +7.0*KDEL2/(256.0*TAU4*TAUSR))
      DA0=A2*C11SQ
      DA1=1.5*A3*C11SQ
      DA2=2.0*A4*C11SQ
      DA4=3.0*A6*C11SQ
      DA5=3.5*A7*C11SQ
      DA6=4.0*A8*C11SQ
      DA8=5.0*A10*C11SQ
      DA9=5.5*A11*C11SQ
      D0=0.5*B1*C11SQ
      D1=1.5*B3*C11SQ
      D3=2.5*B5*C11SQ
      D5=3.5*B7*C11SQ
      D7=4.5*B9*C11SQ
      D9=5.5*B11*C11SQ
C****
C     ARE MODAL EIGENVALUES TO BE CALCULATED OR ARE THEY ALREADY
C     STORED ON FILE
C
C
C     MFILE - IF MFILE=0 CALCULATE EIGENVALUES
C             IF MFILE=1 READ EIGENVALUES FROM FILE 12
C
      IF(MFILE.EQ.1) THEN
      READ(12,971) NRMODE
      DO 90 KQ=1,NRMODE
      READ(12,973) ZEROS(KQ)
```

```
      90    CONTINUE
            GO TO 520
            END IF
C*****
C     SET UP SEARCH RECTANGLES FOR FINDING MODES AND SOLVE FOR MODES
C
C
C     NLEFT = NUMBER OF CONSECUTIVE SEARCH RECTANGLES WITH NO ZEROS TO
C        BE FOUND TO THE LEFT OF THE ORIGINAL SEARCH RECTANGLE BEFORE
C        STOPPING SEARCH TO THE LEFT.
C     NRIGHT = NUMBER OF CONSECUTIVE SEARCH RECTANGLES WITH NO ZEROS TO
C        BE FOUND TO THE RIGHT OF THE ORIGINAL SEARCH RECTANGLE BEFORE
C        STOPPING SEARCH TO THE RIGHT.
C
            NLEFT=5
            NRIGHT=5
C
C     IF INITIALIZE 'ILFLAG=NLEFT' PROGRAM WILL NOT SEARCH TO THE LEFT
C        OF THE ORIGINAL SEARCH RECTANGLE
C     IF INITIALIZE 'IRFLAG=NRIGHT' PROGRAM WILL NOT SEARCH TO THE RIGHT
C        OF THE ORIGINAL SEARCH RECTANGLE
C
            ILFLAG=0
            IRFLAG=0
C
C     TOL - TOLERANCE TO WHICH MODES ARE TO BE FOUND
C     TMESH - MESH SIZE USED BY ROOT FINDING SUBROUTINE 'FZEROX'
C     MPRINT - MPRINT=0 GIVES NO 'DEBUGGING' PRINTOUT FROM
C        SUBROUTINE 'FZEROX' WHILE MPRINT=1 DOES GIVE 'DEBUGGING'
C        PRINTOUT
C     TLOLD - LEFT COORDINATE OF INITIAL SEARCH RECTANGLE
C     TROLD - RIGHT COORDINATE OF INITIAL SEARCH RECTANGLE
C     TBOT - BOTTOM COORDINATE OF SEARCH RECTANGLES
C     TTOP - TOP COORDINATE OF SEARCH RECTANGLES
C     TLSTEP - SIZE OF SEARCH RECTANGLES TO THE LEFT OF THE
C        INITIAL SEARCH RECTANGLE
C     TRSTEP - SIZE OF SEARCH RECTANGLES TO THE RIGHT OF THE
C        INITIAL SEARCH RECTANGLE
C
C     ***
C     IN SUBROUTINE 'FZEROX' THE ERROR CONDITION 'X MODES FOUND
C     ON SAME PHASE LINE' WITH X.GT.1 CAN OFTEN BE CORRECTED BY
C     REDUCING BOTH TMESH AND THE SIZE OF THE SEARCH RECTANGLE.
C     SUBROUTINE 'FZEROX' WILL AUTOMATICALLY REDUCE TMESH BY
C     2.0 TWICE BUT IT DOES NOT REDUCE THE SIZE OF THE
C     SEARCH RECTANGLE.
C     ***
C
            TOL =0.0001
            TMESH=0.04
            MPRINT=0
            NRMODE=0
            TLOLD=0.0
            TROLD=2.0
            TBOT=-TOL
```

```
      TTOP=2.0*(ALOSS*1.0E-3/(WAVENO*C20LOG)+ETA/2.0)/(C11*C11)
      TLEFT=TLOLD
      TRIGHT=TROLD
      TLSTEP=2.0
      TRSTEP=2.0
C
C     THE FLAG 'INTCHK' IS USED TO KEEP TRACK OF WHERE THE CURRENT
C     SEARCH RECTANGLE IS IN RELATION TO THE INITIAL SEARCH
C     RECTANGLE
C
C        INTCHK=0  -  INITIAL SEARCH RECTANGLE
C        INTCHK=1  -  SEARCH RECTANGLE TO THE LEFT OF INITIAL SEARCH
C                     RECTANGLE
C        INTCHK=2  -  SEARCH RECTANGLE TO THE RIGHT OF INITIAL SEARCH
C                     RECTANGLE
C
C     ***
C     NOTE 1: BECAUSE INTCHK IS USED IN SUBROUTINE KHATX TO FORCE THE
C             EVALUATION OF KHAT(Q11) TO BE ON THE 'CORRECT BRANCH'
C             WHEN DELTA IS NOT EQUAL TO ZERO, THE VALUE OF TLOLD
C             MUST EQUAL 0.0
C
C     NOTE 2: AFTER THE MODE SEARCH IS COMPLETED INTCHK IS AGAIN SET
C             EQUAL TO ZERO. THIS IS AGAIN FOR USE IN SUBROUTINE
C             KHATX WHEN EVALUATING KHAT(Q11) IN THE HEIGHT-GAIN
C             CALCULATIONS.
C     ***
C
      INTCHK=0
C
C     CALL ROOT FINDING ROUTINE OF MORFITT AND SHELLMAN TO FIND ZEROS
C     IN CURRENT SEARCH RECTANGLE
C
  210 CALL FZEROX(TLEFT,TRIGHT,TROT,TTOP,TMESH,TOL,MPRINT,EIGEN,NRE)
      WRITE(11,800) TLEFT,TRIGHT,TBOT,TTOP
      IF(NRE.GT.0) THEN
      DO 300 LK=1,NRE
C
C     THE FOLLOWING STATEMENT COMPENSATES FOR THE SHIFT IN Q11
C     INTRODUCED IN SUBROUTINES 'FDFDTX' AND 'FCTVLX' IN ORDER TO AVOID
C     HAVING A MESH LINE ON THE REAL AXIS
C
      EIGEN(LK)=EIGEN(LK)-I*0.003
C
C     ***
C     IF THE VALUE OF TMESH IS CHANGED THE OFFSET IN THE
C     VALUE OF Q11 IN THIS STATEMENT AND IN THE CORRESPONDING
C     STATEMENTS IN SUBROUTINES 'FDFDTX' AND 'FCTVLX' MAY
C     ALSO HAVE TO BE CHANGED IN ORDER THAT A MESH LINE
C     DOES NOT FALL ON THE REAL AXIS
C     SUBROUTINE 'EXCFAC' ALSO HAS A STATEMENT THAT DEPENDS UPON THE
C     SIZE OF THIS OFFSET
C     ***
C
      WRITE(11,801) LK,EIGEN(LK)
```

```
  300 CONTINUE
      END IF
      IF(NRE.EQ.0) GO TO 100
C
C     SINCE SUBROUTINE 'FZEROX' EXTENDS THE SEARCH RECTANGLE
C     SLIGHTLY, CHECK FOR AND ELIMINATE ZEROS FOUND OUTSIDE CURRENT
C     SEARCH RECTANGLE
C
      JKFLAG=0
      DO 150 JK=1,NRE
      IF((REAL(EIGEN(JK-JKFLAG)).GE.TRIGHT).OR.(REAL(EIGEN(JK-JKFLAG)).
     $LT.TLEFT).OR.(AIMAG(EIGEN(JK-JKFLAG)).LT.TBOT).OR.(AIMAG(EIGEN(JK
     $-JKFLAG)).GT.TTOP)) THEN
      JSTOP = NRE-1
      JSTART= JK-JKFLAG
      DO 160 J=JSTART,JSTOP
      EIGEN(J) = EIGEN(J+1)
  160 CONTINUE
      NRE=NRE-1
      JKFLAG=JKFLAG+1
      END IF
  150 CONTINUE
C
C     STORE ZEROS FOUND IN CURRENT SEARCH RECTANGLE IN ARRAY
C     CONTAINING ALL ZEROS.
C
      IF(NRE.GT.0) KN=NRMODE+NRE
      IF(KN.GT.NRZ) THEN
      NROVER=KN-NRZ
      KN=NRZ
      NRMODE=NRZ-NRE
      WRITE(11,902) TLEFT,TRIGHT,NROVER
      END IF
      IF(KN.LT.NRZ) THEN
      JEND=NRE
      ELSE
      JEND=NRE-NROVER
      END IF
      DO 190 J=1,JEND
      ZEROS(NRMODE+J)=EIGEN(J)
  190 CONTINUE
      NRMODE=NRMODE+NRE
C
C     IF 'ILFLAG.NE.NLEFT' FORM NEW SEARCH RECTANGLE TO THE LEFT OF
C     TOTAL REGION SEARCHED.
C
  100 IF(ILFLAG.EQ.NLEFT) GO TO 400
      IF((NRE.EQ.0).AND.(INTCHK.EQ.1)) THEN
      ILFLAG=ILFLAG+1
      ELSE
      ILFLAG=0
      END IF
      IF(ILFLAG.EQ.NLEFT) GO TO 400
      INTCHK=1.
      TLEFT=TLOLD-TLSTEP
```

```
441         TRIGHT=TLOLD
442         TLOLD=TLEFT
443         GO TO 210
444 C
445 C       IF 'IRFLAG.NE.NRIGHT' FORM NEW SEARCH RECTANGLE TO THE RIGHT OF
446 C       TOTAL REGION SEARCHED.
447 C
448    400  IF(IRFLAG.EQ.NRIGHT) GO TO 200
449         IF((NRF.EQ.0).AND.(INTCHK.EQ.2)) THEN
450         IRFLAG=IRFLAG+1
451         ELSE
452         IRFLAG=0
453         END IF
454         IF(IRFLAG.EQ.NRIGHT) GO TO 200
455         INTCHK=2
456         TLEFT=TROLD
457         TRIGHT=TROLD+TRSTEP
458         TROLD=TRIGHT
459         GO TO 210
460 C****
461 C       MODE SEARCH COMPLETED
462 C
463 C
464 C       ORDER ZEROS FOUND BY ORDER OF INCREASING REAL PART.
465 C
466    200  IF(NRMODE.GT.1) THEN
467         JKEND=NRMODE-1
468         DO 410 JK=1,JKEND
469         NEND=NRMODE-JK
470         DO 420 J=1,NEND
471         IF(REAL(ZEROS(NRMODE+1-J)).LT.REAL(ZEROS(NRMODE-J))) THEN
472         CTEMP=ZEROS(NRMODE+1-J)
473         ZEROS(NRMODE+1-J)=ZEROS(NRMODE-J)
474         ZEROS(NRMODE-J)=CTEMP
475         END IF
476    420  CONTINUE
477    410  CONTINUE
478         END IF
479 C
480 C       THE POSSIBILITY EXISTS THAT DUPLICATE (WITHIN THE TOLERANCE 'TOL')
481 C       ZEROS OF THE MODAL EQUATION WILL BE FOUND.  ELIMINATE THESE
482 C       DUPLICATE ZEROS.
483 C
484         JKFLAG=0
485         JKEND=NRMODE-1
486         DO 240 JK=1,JKEND
487         CHKR=ABS(REAL(ZEROS(JK-JKFLAG))-REAL(ZEROS(JK+1-JKFLAG)))
488         CHKI=ABS(AIMAG(ZEROS(JK-JKFLAG))-AIMAG(ZEROS(JK+1-JKFLAG)))
489         CHKSQ=SQRT(CHKR*CHKR+CHKI*CHKI)
490         IF(CHKSQ.LT.TOL) THEN,
491         JSTOP=NRMODE-1
492         JSTART=JK-JKFLAG
493         DO 250 J=JSTART,JSTOP
494         ZEROS(J)=ZEROS(J+1)
495    250  CONTINUE
```

```
      NRMODE=NRMODE-1
      JKFLAG=JKFLAG+1
      END IF
  240 CONTINUE
C
C     FOR VERY WELL TRAPPED MODES THE IMAGINARY PART OF THE ZEROS
C     OF THE MODAL EQUATION MAY BE FOUND TO BE NEGATIVE (BUT LESS
C     IN ABSOLUTE VALUE THAN 'TOL'). SET THESE IMAGINARY PARTS
C     EQUAL TO ZERO
C
      DO 405 J=1,NRMODE
      IF(AIMAG(ZEROS(J)).LT.0.0) ZEROS(J)=ONE*REAL(ZEROS(J))
  405 CONTINUE
C
C     CONVERT MODES IN COMPLEX G11-PLANE TO MODES IN THE COMPLEX
C     THETA PLANE. THIS OUTPUT IS FOR COMPARISON PURPOSES ONLY.
C
C     ANGLES ARE REFERENCED TO THE HEIGHT ZREF WHERE M(Z)=M(ZREF)
C     IS SPECIFIED.
C
      DO 220 J=1,NRMODE
      RAD=CSQRT(ZEROS(J)*C11*C11+ALPHA1*H)
      THETA(J)=CASIN(RAD)
  220 CONTINUE
C
C     CALCULATE ATTENUATION RATE FOR DIFFERENT MODES AND OUTPUT RESULTS
C
      WRITE(11,903)
      IF(NRMODE.GT. 0.0) THEN
      WRITE(12,972) NRMODE
      DO 230 J=1,NRMODE
      BETASQ=1.0*I*ETAABS-ALPHA1*H-C11*C11*ZEROS(J)
      BETAI=AIMAG(CSQRT(BETASQ))
      ATNU=-C20LOG*WAVENO*BETAI*1.0E+3
      WRITE(11,904) J,ZEROS(J),J,THETA(J),ATNU
      WRITE(12,970) J,ZEROS(J)
  230 CONTINUE
      END IF
C****
C     CALCULATE FIELD STRENGTH RELATIVE TO FREE SPACE VALUE
C
  520 WRITE(11,905)
      WRITE(11,907)
      INTCHK=0
      DO 505 JRNG=1,NRNG
      RNG=RNGINT+FLOAT(JRNG-1)*RNGSTP
      DO 500 JXMTR=1,NZXMTR
      ZXMTR=ZXINIT+FLOAT(JXMTR-1)*ZXSTEP
      DO 510 JRCVR=1,NZRCVR
      ZRCVR=ZRINIT+FLOAT(JRCVR-1)*ZRSTEP
      CALL MODSUM(ECMS,EIMS,RNG,ZRCVR,ZXMTR,ZEROS,NRZ,NRMODE)
      CALL DHORIZ(DHZ,ZRCVR,ZXMTR)
      WRITE(11,908) RNG,ZXMTR,ZRCVR,ECMS,EIMS,DHZ
  510 CONTINUE
  500 CONTINUE
```

```
  505 CONTINUE
C****
      STOP
      END
```

```
      SUBROUTINE CADD(ZA,ZE,XA,XE,YA,YE)
C**********
C     THIS SUBROUTINE TAKES THE EXTENDED COMPLEX NUMBERS 'X' AND 'Y'
C     AND ADDS THEM TO FORM THE EXTENDED COMPLEX NUMBER 'Z'
C
C     THE EXTENDED COMPLEX NUMBER 'Z' IS MADE UP OF THE COMPLEX
C     AMPLITUDE 'ZA' AND THE REAL EXPONENT 'ZE'. THUS, 'Z' IS
C     EQUAL TO 'Z=ZA*EXP(ZE)'
C**********
      IMPLICIT DOUBLE PRECISION*8 (A-H,O-Z)
      COMPLEX ZA,XA,YA
      IF((REAL(XA).EQ.0.0).AND.(AIMAG(XA).EQ.0.0)) THEN
        IF((REAL(YA).EQ.0.0).AND.(AIMAG(YA).EQ.0.0)) THEN
          ZA=(0.0,0.0)
          ZE=0.0
          RETURN
        ELSE
          ZA=YA
          ZE=YE
          RETURN
        END IF
      END IF
      IF((REAL(YA).EQ.0.0).AND.(AIMAG(YA).EQ.0.0)) THEN
        ZA=XA
        ZE=XE
        RETURN
      END IF
      E=YE-XE
      IF(XE.GE.YE) THEN
        IF(E.GT.-85.0) THEN
          R=EXP(E)
        ELSE
          R=0.0
        END IF
        ZA=XA+R*YA
        ZE=XE
      ELSE
        IF(-E.GT.-85.0) THEN
          R=EXP(-E)
        ELSE
          R=0.0
        END IF
        ZA=YA+R*XA
        ZE=YE
      END IF
      IF((REAL(ZA).EQ.0.0).AND.(AIMAG(ZA).EQ.0.0))  ZE=0.0
      RETURN
      END
```

```
1         SUBROUTINE ADDX(ZA,ZE,XA,XE,YA,YE)
2  C**********
3  C      THIS PROGRAM TAKES THE EXTENDED REAL NUMBERS 'X' AND 'Y'
4  C      AND ADDS THEM TO FORM THE EXTENDED REAL NUMBER 'Z'
5  C
6  C      THE EXTENDED REAL NUMBER 'Z' IS MADE UP OF THE
7  C      AMPLITUDE 'ZA' AND THE EXPONENT 'ZE'.  'Z' IS EQUAL TO
8  C      'Z=ZA*EXP(ZE)'
9  C**********
10         IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
11         IF(XA.EQ.0.0) THEN
12           IF(YA.EQ.0.0) THEN
13             ZA=0.0
14             ZE=0.0
15             RETURN
16           ELSE
17             ZA=YA
18             ZE=YE
19             RETURN
20           END IF
21         END IF
22         IF(YA.EQ.0.0) THEN
23           ZA=XA
24           ZE=XE
25           RETURN
26         END IF
27         E=YE-XE
28         IF(XE.GE.YE) THEN
29           IF(E.GT.-85.0) THEN
30             R=EXP(E)
31           ELSE
32             R=0.0
33           END IF
34           ZA=XA+R*YA
35           ZE=XE
36         ELSE
37           IF(-E.GT.-85.0) THEN
38             R=EXP(-E)
39           ELSE
40             R=0.0
41           END IF
42           ZA=YA+R*XA
43           ZE=YE
44         END IF
45         IF(ZA.EQ.0.0) ZE=0.0
46         RETURN
47         END
```

```
1       SUBROUTINE HNKLX(Z,H1A,H1E,H2A,H2E,DH1A,DH1E,DH2A,DH2E)
2 C**********
3 C     THIS PROGRAM EVALUATES THE MODIFIED HANKEL FUNCTIONS H1(Z) AND
4 C     H2(Z) AND THEIR DERIVATIVES H1'(Z) AND H2'(Z) WITH A 'RELATIVE
5 C     ACCURACY' BETTER THAN 10**(-3)
6 C
7 C     THE VALUES ARE RETURNED AS A COMPLEX AMPLITUDE (E.G. H1A)
8 C     AND A REAL EXPONENT(E.G. H1E). THE MODIFIED HANKEL FUNCTION
9 C     H1(Z) WOULD THEN EQUAL H1(Z)=H1A*EXP(H1E)
10 C**********
11       IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
12       DOUBLE PRECISION*6  IMZ,IMX1
13       COMPLEX  I,CON1,CON2,CON2CJ,CE1,CE1CJ,CE2,CE2CJ,Z,ZSQ,ZCUBE,
14      *   SUMFZ,SUMGZ,SUMFPZ,SUMGPZ,FZ,GZ,FPZ,GPZ,H1A,H2A,DH1A,DH2A,
15      *   CONFPZ,ZSQR,CSQRT,ZFR,XI,CEXP,ZTH,ZMTH,EXI,SUMCKP,SUMCKM,
16      *   SUMDKP,SUMDKM,CONFZ
17       DIMENSION A(9),B(9),C(9),D(9),CK(5),DK(5)
18 C**   ARRAY CONTAINING COEFFICIENTS FOR F(Z)
19       DATA (A(N),N=1,9)/ -1.66666667E-1, +5.55555556E-3, -7.71604938E-5,
20      *   +5.84549196E-7, -2.78356760E-9, +9.09662614E-12,
21      *   -2.16586337E-14, +3.92366552E-17, -5.58926712E-20/
22 C**   ARRAY CONTAINING COEFFICIENTS FOR G(Z)
23       DATA (B(N),N=1,9)/ -8.33333333E-2, +1.98412698E-3, -2.20458554E-5,
24      *   +1.41319586E-7, -5.88831607E-10, +1.72172985E-12,
25      *   -3.72668798E-15, +6.21114663E-18, -8.21580242E-21/
26 C**   ARRAY CONTAINING COEFFICIENTS FOR F'(Z)
27       DATA (C(N),N=1,9)/ -6.66666667E-2, +1.38888889E-3, -1.40291807E-5,
28      *   +8.35070280E-8, -3.27478541E-10, +9.09662614E-13,
29      *   -1.88335945E-15, +3.01820425E-18, -3.85466698E-21/
30 C**   ARRAY CONTAINING COEFFICIENTS FOR G'(Z)
31       DATA (D(N),N=1,9)/ -3.33333333E-1, +1.38888889E-2, -2.20458554E-4,
32      *   +1.83715461E-6, -9.42130572E-9, +3.27128671E-11,
33      *   -8.19871355E-14, +1.55278666E-16, -2.30042468E-19/
34 C**   ARRAY CONTAINING COEFFICIENTS IN ASYMPTOTIC EXPANSION OF H1(Z)
35 C     AND H2(Z)
36       DATA (CK(N),N=1,5)/ 1.04166667E-1, 8.35503472E-2, 1.28226575E-1,
37      *   2.91849026E-1, 8.81627267E-1/
38 C**   ARRAY CONTAINING COEFFICIENTS IN ASYMPTOTIC EXPANSION OF H1'(Z)
39 C     AND H2'(Z)
40       DATA (DK(N),N=1,5)/ +1.45833333E-1, -9.87413194E-2,
41      *   +1.43312054E-1, -3.17227203E-1, +9.42429148E-1/
42 C
43       DATA X0,X1,ALPHA,PI,PI3/ 9.30436717E-1, 6.78298725E-1,
44      *   8.53667219E-1, 3.14159265E0, 1.04719755E0/
45 C
46       DATA I,CON1,CON2,CON2CJ/ (0.0,1.0), (0.0,1.15470054),
47      *   (1.0,0.577350269), (1.0,-0.577350269)/
48 C
49       DATA  CE1,CE1CJ,CE2,CE2CJ/ (0.258819045,0.965925826),
50      *   (0.258819045,-0.965925826), (-0.965925826,0.258819045),
51      *   (-0.965925826,-0.258819045)/
52 C****
53 C     TEST WHETHER TO EVALUATE MODIFIED HANKEL FUNCTIONS BY SERIES
54 C     EXPANSION OR ASYMPTOTIC EXPANSION
55 C
```

```
      AMPZ=CABS(Z)
      IF(AMPZ.LT.(3.5)) THEN
C**********
C     EVALUATE MODIFIED HANKEL FUNCTIONS USING SERIES EXPANSIONS
C
      ZSQ=Z*Z
      ZCUBE=Z*ZSQ
      SUMFZ=A(9)*ZCUBE
      SUMGZ=B(9)*ZCUBE
      SUMFPZ=C(9)*ZCUBE
      SUMGPZ=D(9)*ZCUBE
      DO 100 K=8,1,-1
      SUMFZ=(SUMFZ+A(K))*ZCUBE
      SUMGZ=(SUMGZ+B(K))*ZCUBE
      SUMFPZ=(SUMFPZ+C(K))*ZCUBE
      SUMGPZ=(SUMGPZ+D(K))*ZCUBE
  100 CONTINUE
      FZ=(SUMFZ+1.0)*X0
      GZ=(SUMGZ+1.0)*X1*Z
      FPZ=-(SUMFPZ+1.0)*Y0*ZSQ/2.0
      GPZ=(SUMGPZ+1.0)*X1
      CONFZ=CON1*FZ
      CONFPZ=CON1*FPZ
C
      K1A=CON2*GZ-CONFZ
      K1E=0.0
      K2A=CON2*(GZ+FZ)
      K2E=0.0
      DK1A=CON2*GPZ-CONFPZ
      DK1E=0.0
      DK2A=CON2*(GPZ+FPZ)
      DK2E=0.0
      RETURN
C**********
      ELSE
C**********
C     EVALUATE MODIFIED HANKEL FUNCTIONS USING ASYMPTOTIC EXPANSIONS
C
      ZSQR=CSQRT(Z)
      ZFR=CSQRT(ZSQR)
      ZTH=Z*ZSQR
      ZMTH=1/ZTH
      XI=2.0*ZTH/3.0
      REXI=REAL(XI)
       IMXI=AIMAG(XI)
       EXI=CEXP(I*REXI)
       SUMCKP=CK(5)*ZMTH
       SUMCKM=-SUMCKP
       SUMDKP=DK(5)*ZMTH
       SUMDKM=-SUMDKP
       DO 102 K=4,1,-1
       SUMCKP=(SUMCKP+CK(K))*ZMTH
       SUMCKM=-(SUMCKM+CK(K))*ZMTH
       SUMDKP=(SUMDKP+DK(K))*ZMTH
       SUMDKM=-(SUMDKM+DK(K))*ZMTH
```

```
  102 CONTINUE
      SUMCKP=1.0+SUMCKP
      SUMCKM=1.0+SUMCKM
      SUMDKP=1.0+SUMDKP
      SUMDKM=1.0+SUMDKM
C
C     EVALUATE PH(Z)
C
      REZ=REAL(Z)
      IMZ=AIMAG(Z)
      IF(REZ.GT.(0.0)) THEN
      PHZ=ATAN(IMZ/REZ)
      ELSE IF(REZ.LT.(0.0)) THEN
      IF(IMZ.GE.(0.0)) THEN
      PHZ=PI+ATAN(IMZ/REZ)
      ELSE
      PHZ=-PI+ATAN(IMZ/REZ)
      END IF
      ELSE
      IF(IMZ.GT.(0.0)) THEN
      PHZ=PI/2.0
      ELSE IF(IMZ.LT.(0.0)) THEN
      PHZ=-PI/2.0
      ELSE
      PHZ=0.0
      END IF
      END IF
C
C     EVALUATE MODIFIED HANKEL FUNCTIONS
C
      IF(PHZ.GE.-PI3) THEN
      H1A=ALPHA*EXI*CE1CJ*SUMCKM/ZFR
      H1E=-IMXI
      IF((REAL(H1A).EQ.0.0).AND.(AIMAG(H1A).EQ.0.0))  H1E=0.0
      DH1A=ALPHA*I*ZFR*EXI*CE1CJ*SUMDKP
      DH1E=-IMXI
      IF((REAL(DH1A).EQ.0.0).AND.(AIMAG(DH1A).EQ.0.0))  DH1E=0.0
      ELSE
      IF(IMXI.GE.0.0) THEN
      ESM=-2.0*IMXI
      IF(ESM.GT.-85.0) THEN
      REEXP=EXP(ESM)
      ELSE
      REEXP=0.0
      END IF
      H1A=ALPHA*(CE2CJ*SUMCKP/EXI+REEXP*EXI*CE1CJ*SUMCKM)/ZFR
      H1E=IMXI
      IF((REAL(H1A).EQ.0.0).AND.(AIMAG(H1A).EQ.0.0))  H1E=0.0
      DH1A=ALPHA*I*ZFR*(REEXP*EXI*CE1CJ*SUMDKP-CE2CJ*SUMDKM/EXI)
      DH1E=IMXI
      IF((REAL(DH1A).EQ.0.0).AND.(AIMAG(DH1A).EQ.0.0))  DH1E=0.0
      ELSE
      ESM=2.0*IMXI
      IF(ESM.GT.-85.0) THEN
      REEXP=EXP(ESM)
```

```
      ELSE
      REEXP=0.0
      END IF
      H1A=ALPHA*(REEXP*CE2CJ*SUMCKP/EXI+EXI*CE1CJ*SUMCKM)/ZFR
      H1F=-IMXI
      IF((REAL(H1A).EQ.0.0).AND.(AIMAG(H1A).EQ.0.0))  H1E=0.0
      DH1A=ALPHA*I*ZFR*(EXI*CE1CJ*SUMDKP-REEXP*CE2CJ*SUMDKM/EXI)
      DH1E=-IMXI
      IF((REAL(DH1A).EQ.0.0).AND.(AIMAG(DH1A).EQ.0.0))  DH1E=0.0
      END IF
      END IF
      IF(PHZ.GT.PI3) THEN
      IF(IMXI.GE.0.0) THEN
      ESM=-2.0*IMXI
      IF(ESM.GT.-85.0) THEN
      REEXP=EXP(ESM)
      ELSE
      REEXP=0.0
      END IF
      H2A=ALPHA*(CE1*SUMCKP/EXI+REEXP*EXI*CE2*SUMCKM)/ZFR
      H2F=IMXI
      IF((REAL(H2A).EQ.0.0).AND.(AIMAG(H2A).EQ.0.0))  H2E=0.0
      DH2A=ALPHA*I*ZFR*(REEXP*EXI*CE2*SUMDKP-CE1*SUMDKM/EXI)
      DH2E=IMXI
      IF((REAL(DH2A).EQ.0.0).AND.(AIMAG(DH2A).EQ.0.0))  DH2E=0.0
      ELSE
      ESM=2.0*IMXI
      IF(ESM.GT.-85.0) THEN
      REEXP=EXP(ESM)
      ELSE
      REEXP=0.0
      END IF
      H2A=ALPHA*(REEXP*CE1*SUMCKP/EXI+EXI*CE2*SUMCKM)/ZFR
      H2E=-IMXI
      IF((REAL(H2A).EQ.0.0).AND.(AIMAG(H2A).EQ.0.0))  H2E=0.0
      DH2A=ALPHA*I*ZFR*(EXI*CE2*SUMDKP-REEXP*CE1*SUMDKM/EXI)
      DH2E=-IMXI
      IF((REAL(DH2A).EQ.0.0).AND.(AIMAG(DH2A).EQ.0.0))  DH2E=0.0
      END IF
      ELSE
      H2A=ALPHA*CE1*SUMCKP/(ZFR*EXI)
      H2F=IMXI
      IF((REAL(H2A).EQ.0.0).AND.(AIMAG(H2A).EQ.0.0))  H2E=0.0
      DH2A=-ALPHA*I*ZFR*CE1*SUMDKM/EXI
      DH2E=IMXI
      IF((REAL(DH2A).EQ.0.0).AND.(AIMAG(DH2A).EQ.0.0))  DH2E=0.0
      END IF
      END IF
      RETURN
C**********
      END
```

```
      SUBROUTINE KHNKLX(Z,K1A,K1E,K2A,K2E,DK1A,DK1E,DK2A,DK2E)
C**********
C     THIS PROGRAM EVALUATES THE MODIFIED HANKEL FUNCTIONS K1(Z) AND
C     K2(Z) AND THEIR DERIVATIVES K1'(Z) AND K2'(Z) WITH A 'RELATIVE
C     ERROR' BETTER THAN 10**(-3)
C
C     THE VALUES ARE RETURNED AS A COMPLEX AMPLITUDE (E.G. K1A)
C     AND A REAL EXPONENT (E.G. K1E). THE MODIFIED HANKEL FUNCTION
C     K1(Z) WOULD THEN EQUAL K1(Z)=K1A*EXP(K1E)
C**********
       IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
       DOUBLE PRECISION*6 IMZ,K1E,K2E,IMXI
       COMPLEX   I,CON1,CON2,CON2CJ,CE1,CE1CJ,CE2,CE2CJ,Z,ZSQ,ZCUBE,
     $   SUMFZ,SUMGZ,SUMFPZ,SUMGPZ,FZ,GZ,FPZ,GPZ,K1A,K2A,DK1A,DK2A,
     $   CONFPZ,ZSQR,CSQRT,ZFR,XI,CEXP,ZTH,ZMTH,EXI,SUMCKP,SUMCKM,
     $   SUMDKP,SUMDKM,CONFZ
       DIMENSION A(9),B(9),C(9),D(9),CK(5),DK(5)
C**    ARRAY CONTAINING COEFFICIENTS FOR F(Z)
       DATA (A(N),N=1,9)/ -1.66666667E-1, +5.55555556E-3, -7.71604938E-5,
     $   +5.84549196E-7, -2.78356760E-9, +9.09662614E-12,
     $   -2.16586337E-14, +3.92366552E-17, -5.58926712E-20/
C**    ARRAY CONTAINING COEFFICIENTS FOR G(Z)
       DATA (B(N),N=1,9)/ -8.33333333E-2, +1.98412698E-3, -2.20458554E-5,
     $   +1.41319586E-7, -5.88831607E-10, +1.72172985E-12,
     $   -3.72668798E-15, +6.21114663E-18, -8.21580242E-21/
C**    ARRAY CONTAINING COEFFICIENTS FOR F'(Z)
       DATA (C(N),N=1,9)/ -6.66666667E-2, +1.38888889E-3, -1.40291807E-5,
     $   +8.35070280E-8, -3.27478541E-10, +9.09662614E-13,
     $   -1.88335945E-15, +3.01820425E-18, -3.85466698E-21/
C**    ARRAY CONTAINING COEFFICIENTS FOR G'(Z)
       DATA (D(N),N=1,9)/ -3.33333333E-1, +1.38888889E-2, -2.20458554E-4,
     $   +1.83715461E-6, -9.42130572E-9, +3.27128671E-11,
     $   -8.19871355E-14, +1.55278666E-16, -2.30042468E-19/
C**    ARRAY CONTAINING COEFFICIENTS IN ASYMPTOTIC EXPANSION OF K1(Z)
C      AND K2(Z)
       DATA (CK(N),N=1,5)/ 1.04166667E-1, 8.35503472E-2, 1.28226575E-1,
     $   2.91849026E-1, 8.81627267E-1/
C**    ARRAY CONTAINING COEFFICIENTS IN ASYMPTOTIC EXPANSION OF K1'(Z)
C      AND K2'(Z)
       DATA (DK(N),N=1,5)/ +1.45833333E-1, -9.87413194E-2,
     $   +1.43312054E-1, -3.17227203E-1, +9.42429148E-1/
C
       DATA X0,X1,ALPHA,PI,PI3/ 9.30436717E-1, 6.78298725E-1,
     $   8.53667219E-1, 3.14159265E0, 1.04719755E0/
C
       DATA I,CON1,CON2,CON2CJ/ (0.0,1.0), (0.0,1.15470054),
     $   (1.0,0.577350269), (1.0,-0.577350269)/
C
       DATA  CE1,CE1CJ,CE2,CE2CJ/ (0.258819045,0.965925826),
     $   (0.258819045,-0.965925826), (-0.965925826,0.258819045),
     $   (-0.965925826,-0.258819045)/
C****
C      TEST WHETHER TO EVALUATE MODIFIED HANKEL FUNCTIONS BY SERIES
C      EXPANSION OR ASYMPTOTIC EXPANSION
C
```

```
      AMPZ=CABS(Z)
      IF(AMPZ.LT.(3.5)) THEN
C**********
C     EVALUATE MODIFIED HANKEL FUNCTIONS USING SERIES EXPANSIONS
C
      ZSQ=Z*Z
      ZCUBE=Z*ZSQ
      SUMFZ=A(9)*ZCUBE
      SUMGZ=B(9)*ZCUBE
      SUMFPZ=C(9)*ZCUBE
      SUMGPZ=D(9)*ZCUBE
      DO 100 K=8,1,-1
      SUMFZ=(SUMFZ+A(K))*ZCUBE
      SUMGZ=(SUMGZ+B(K))*ZCUBE
      SUMFPZ=(SUMFPZ+C(K))*ZCUBE
      SUMGPZ=(SUMGPZ+D(K))*ZCUBE
  100 CONTINUE
      FZ=(SUMFZ+1.0)*X0
      GZ=(SUMGZ+1.0)*X1*Z
      FPZ=-(SUMFPZ+1.0)*X0*ZSQ/2.0
      GPZ=(SUMGPZ+1.0)*X1
      CONFZ=CON1*FZ
      CONFPZ=CON1*FPZ
C
      H1A=CON2*GZ-CONFZ
      H1E=0.0
      H2A=CON2CJ*GZ+CONFZ
      H2E=0.0
      DH1A=CON2*GPZ-CONFPZ
      DH1E=0.0
      DH2A=CON2CJ*GPZ+CONFPZ
      DH2E=0.0
      RETURN
C**********
      ELSE
C**********
C     EVALUATE MODIFIED HANKEL FUNCTIONS USING ASYMPTOTIC EXPANSIONS
C
      ZSQR=CSQRT(Z)
      ZFR=CSQRT(ZSQR)
      ZTH=Z*ZSQR
      ZMTH=1/ZTH
      XI=2.0*ZTH/3.0
      REXI=REAL(XI)
       IMXI=AIMAG(XI)
       EXI=CEXP(I*REXI)
       SUMCKP=CK(5)*ZMTH
       SUMCKM=-SUMCKP
       SUMDKP=DK(5)*ZMTH
       SUMDKM=-SUMDKP
       DO 102 K=4,1,-1
       SUMCKP=(SUMCKP+CK(K))*ZMTH
       SUMCKM=-(SUMCKM+CK(K))*ZMTH
       SUMDKP=(SUMDKP+DK(K))*ZMTH
       SUMDKM=-(SUMDKM+DK(K))*ZMTH
```

```
  102 CONTINUE
      SUMCKP=1.0+SUMCKP
      SUMCKM=1.0+SUMCKM
      SUMDKP=1.0+SUMDKP
      SUMDKM=1.0+SUMDKM
C
C     EVALUATE PH(Z)
C
      REZ=REAL(Z)
      IMZ=AIMAG(Z)
      IF(REZ.GT.(0.0)) THEN
      PHZ=ATAN(IMZ/REZ)
      ELSE IF(REZ.LT.(0.0)) THEN
      IF(IMZ.GE.(0.0)) THEN
      PHZ=PI+ATAN(IMZ/REZ)
      ELSE
      PHZ=-PI+ATAN(IMZ/REZ)
      END IF
      ELSE
      IF(IMZ.GT.(0.0)) THEN
      PHZ=PI/2.0
      ELSE IF(IMZ.LT.(0.0)) THEN
      PHZ=-PI/2.0
      ELSE
      PHZ=0.0
      END IF
      END IF
C
C     EVALUATE MODIFIED HANKEL FUNCTIONS
C
      IF(PHZ.GE.-PI3) THEN
      K1A=ALPHA*EXI*CE1CJ*SUMCKM/ZFR
      K1E=-IMXI
      IF((REAL(K1A).EQ.0.0).AND.(AIMAG(K1A).EQ.0.0))  K1E=0.0
      DK1A=ALPHA*I*ZFR*EXI*CE1CJ*SUMDKP
      DK1E=-IMXI
      IF((REAL(DK1A).EQ.0.0).AND.(AIMAG(DK1A).EQ.0.0))  DK1E=0.0
      ELSE
      IF(IMXI.GE.0.0) THEN
      ESM=-2.0*IMXI
      IF(ESM.GT.-85.0) THEN
      REEXP=EXP(ESM)
      ELSE
      REEXP=0.0
      END IF
      K1A=ALPHA*(CE2CJ*SUMCKP/EXI+EXI*CE1CJ*SUMCKM*REEXP)/ZFR
      K1E=IMXI
      IF((REAL(K1A).EQ.0.0).AND.(AIMAG(K1A).EQ.0.0))  K1E=0.0
      DK1A=ALPHA*I*ZFR*(EXI*CE1CJ*SUMDKP*REEXP-CE2CJ*SUMDKM/EXI)
      DK1E=IMXI
      IF((REAL(DK1A).EQ.0.0).AND.(AIMAG(DK1A).EQ.0.0))  DK1E=0.0
      ELSE
      ESM=2.0*IMXI
      IF(ESM.GT.-85.0) THEN
      REEXP=EXP(ESM)
```

```
      ELSE
      REEXP=0.0
      END IF
      K1A=ALPHA*(CE2CJ*SUMCKP*REEXP/EXI+EXI*CE1CJ*SUMCKM)/ZFR
      K1E=-IMXI
      IF((REAL(K1A).EQ.0.0).AND.(AIMAG(K1A).EQ.0.0))  K1E=0.0
      DK1A=ALPHA*I*ZFR*(EXI*CE1CJ*SUMDKP-CE2CJ*SUMDKM*REEXP/EXI)
      DK1E=-IMXI
      IF((REAL(DK1A).EQ.0.0).AND.(AIMAG(DK1A).EQ.0.0))  DK1E=0.0
      END IF
      END IF
      IF(PHZ.GT.PI3) THEN
      K2A=ALPHA*CE1*SUMCKP/(ZFR*EXI)
      K2E=IMXI
      IF((REAL(K2A).EQ.0.0).AND.(AIMAG(K2A).EQ.0.0))  K2E=0.0
      DK2A=-ALPHA*I*ZFR*CE1*SUMDKM/EXI
      DK2E=IMXI
      IF((REAL(DK2A).EQ.0.0).AND.(AIMAG(DK2A).EQ.0.0))  DK2E=0.0
      ELSE IF(PHZ.LT.-PI3) THEN
      K2A=-ALPHA*EXI*CE2*SUMCKM/ZFR
      K2E=-IMXI
      IF((REAL(K2A).EQ.0.0).AND.(AIMAG(K2A).EQ.0.0))  K2E=0.0
      DK2A=-ALPHA*I*ZFR*EXI*CE2*SUMDKP
      DK2E=-IMXI
      IF((REAL(DK2A).EQ.0.0).AND.(AIMAG(DK2A).EQ.0.0))  DK2E=0.0
      ELSE
      IF(IMXI.GE.0.0) THEN
      ESM=-2.0*IMXI
      IF(ESM.GT.-85.0) THEN
      REEXP=EXP(ESM)
      ELSE
      REEXP=0.0
      END IF
      K2A=ALPHA*(CE1*SUMCKP/EXI-REEXP*CE2*EXI*SUMCKM)/ZFR
      K2E=IMXI
      IF((REAL(K2A).EQ.0.0).AND.(AIMAG(K2A).EQ.0.0))  K2E=0.0
      DK2A=-ALPHA*I*ZFR*(CE1*SUMDKM/EXI+REEXP*EXI*CE2*SUMDKP)
      DK2E=IMXI
      IF((REAL(DK2A).EQ.0.0).AND.(AIMAG(DK2A).EQ.0.0))  DK2E=0.0
      ELSE
      ESM=2.0*IMXI
      IF(ESM.GT.-85.0) THEN
      REEXP=EXP(ESM)
      ELSE
      REEXP=0.0
      END IF
      K2A=ALPHA*(CE1*REEXP*SUMCKP/EXI-EXI*CE2*SUMCKM)/ZFR
      K2E=-IMXI
      IF((REAL(K2A).EQ.0.0).AND.(AIMAG(K2A).EQ.0.0))  K2E=0.0
      DK2A=-ALPHA*I*ZFR*(REEXP*CE1*SUMDKM/EXI+EXI*CE2*SUMDKP)
      DK2E=-IMXI
      IF((REAL(DK2A).EQ.0.0).AND.(AIMAG(DK2A).EQ.0.0))  DK2E=0.0
      END IF
      END IF
      END IF
      RETURN
C**********
      END
```

```
1       SUBROUTINE KHATX(Q11,KHA1,KHE1,KHA2,KHE2,DKHA1,DKHE1,DKHA2,DKHE2)
2 C**********
3 C     THIS SUBROUTINE CALCULATES THE MODIFIED HANKEL FUNCTION
4 C     COMBINATIONS KHAT1(Q11), KHAT2(Q11), AND THEIR DERIVATIVES
5 C
6 C     SUBROUTINES USED...
7 C        KHNKLX,CADD,CTANH,CSECH2
8 C**********
9       IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
10       DOUBLE PRECISION*6 KHE1,KHE2,KDEL2
11       COMPLEX  I,Q11,KHA1,KHA2,DKHA1,DKHA2,Q11A1,Q11A2,DQ11A1,DQ11A2,
12      $     CMU,DCMU,SQNG,BETASQ,FAC,XAA,XBA,XCA,CSQRT,CTANH,CSECH2,
13      $     GAM,HTAN,HSEC2,Q11SR,Z,ZSQ,A,B,DA,DB,A0,A2,A4,A6,A8,A10,
14      $     B1,B3,B5,B7,B9,B11,DA0,DA2,DA4,DA6,DA8,D0,D1,D3,D5,D7,D9
15       EXTERNAL CTANH,CSECH2
16       COMMON/COMTWO/ETAABS,ALPHA1,H,C11,SQNG
17       COMMON/COMTHR/Z2,Z3,A2MAG,ALPHA3,WAVENO,ALPHA2
18       COMMON/COMFOR/DELTA,KDEL2,RNONG
19       COMMON/COMFIV/A0,A2,A3,A4,A6,A7,A8,A10,A11
20       COMMON/COMSIX/B0,B1,B3,B5,B7,B9,B11
21       COMMON/COMSEV/DA0,DA1,DA2,DA4,DA5,DA6,DA8,DA9
22       COMMON/COMEIG/D0,D1,D3,D5,D7,D9
23       COMMON/COMNIN/INTCHK
24       DATA  I/(0.0,1.0)/
25       DATA  SMALL/0.70710687119D-1/
26 C*****
27 C     CALCULATE MODIFIED HANKEL FUNCTIONS NEEDED
28 C
29       CALL KHNKLX(Q11,Q11A1,Q11E1,Q11A2,Q11E2,DQ11A1,DQ11E1,
30      $   DQ11A2,DQ11E2)
31 C*****
32 C     CALCULATE MU(Q11) AND ITS DERIVATIVES
33 C
34 C
35 C     ***
36 C     BECAUSE OF THE DISCONTINUITY IN KHAT(Q11) AT RE(Q11)=0 WHEN
37 C     DELTA IS NOT EQUAL TO ZERO, WHEN DOING THE MODE SEARCH WE
38 C     HAVE TO FORCE KHAT(Q11) TO BE ON THE 'CORRECT BRANCH' WHEN
39 C     SEARCHING THE RECTANGLE WITH LEFT HAND SIDE RE(Q11)=0 AND
40 C     WHEN SEARCHING THE RECTANGLE WITH RIGHT HAND SIDE RE(Q11)=0.
41 C     THE FLAG 'INTCHK' FROM THE MAIN PROGRAM IS USED FOR THIS
42 C     PURPOSE.
43 C     ***
44 C
45       IF(REAL(Q11).GT.0.0) MDEL=1
46       IF((REAL(Q11).EQ.0.0).AND.((INTCHK.EQ.0).OR.(INTCHK.EQ.2))) MDEL=1
47       IF(REAL(Q11).LT.0.0) MDEL=0
48       IF((REAL(Q11).EQ.0.0).AND.(INTCHK.EQ.1)) MDEL=0
49       IF(MDEL.EQ.0) THEN
50 C
51 C
52       BETASQ=1.0+I*ETAABS-ALPHA1*H-C11*C11*Q11
53       FAC=CSQRT(SQNG-BETASQ)
54       CMU=-I*FAC*RNONG/C11
55       DCMU=-I*C11*RNONG/(2.0*FAC)
```

```
      END IF
C***
C
      IF(MDEL.EQ.1) THEN
C
C
      IF(DELTA.EQ.0.0) THEN
      BETASQ=1.0+I*ETAABS-ALPHA1*H-C11*C11*Q11
      FAC=CSQRT(SQNG-BETASQ)
      CMU=-I*FAC*RNONG/C11
      DCMU=-I*C11*RNONG/(2.0*FAC)
C
C
      ELSE
C
C
      Q11SR=CSQRT(Q11)
      GAM=KDEL2*C11*C11*Q11
C
C
      IF((REAL(Q11).LT.SMALL).AND.(AIMAG(Q11).LT.SMALL)) THEN
C
C
      Z=C11*Q11SR
      ZSQ=C11*C11*Q11
      A=A0+ZSQ*(A2+Z*(A3+Z*(A4+ZSQ*(A6+Z*(A7+Z*(A8+ZSQ*(A10+Z*A11)))))))
      B=B0+Z*(B1+ZSQ*(B3+ZSQ*(B5+ZSQ*(B7+ZSQ*(B9+ZSQ*B11)))))
      DA=DA0+Z*(DA1+Z*(DA2+ZSQ*(DA4+Z*(DA5+Z*(DA6+ZSQ*(DA8+Z*DA9))))))
      DB=(D0+ZSQ*(D1+ZSQ*(D3+ZSQ*(D5+ZSQ*(D7+ZSQ*D9)))))/Z
      CMU=-I*A/(B*C11)
      DCMU=(-I/C11)*(DA/B-A*DB/(B*B))
C
C
      ELSE
C
C
      BETASQ=1.0+I*ETAABS-ALPHA1*H-C11*C11*Q11
      FAC=CSQRT(SQNG-BETASQ)
      HTAN=CTANH(GAM)
      A=C11*Q11SR*HTAN+FAC*RNONG
      B=1.0+FAC*RNONG*HTAN/(Q11SR*C11)
      HSEC2=CSECH2(GAM)
      DA=HTAN*C11/(2.0*Q11SR)+KDEL2*ALPHA1*Q11SR*HSEC2/WAVENO
     $   +C11*C11*RNONG/(2.0*FAC)
       DB=RNONG*(-FAC*HTAN/(2.0*C11*Q11*Q11SR)+C11*HTAN/(2.0*FAC*Q11SR)
     $   +C11*FAC*KDEL2*HSEC2/Q11SR)
       CMU=-I*A/(B*C11)
       DCMU=(-I/C11)*(DA/B-A*DB/(B*B))
       END IF
       END IF
       END IF
C*****
C      CALCULATE KHAT1(Q11), KHAT2(Q11), AND THEIR DERIVATIVES
C
       XAA=CMU*Q11A1
```

```
      XAE=Q11E1
      CALL CADD(KHA1,KHE1,DQ11A1,DQ11E1,XAA,XAE)
      XAA=CML*Q11A2
      XAE=Q11E2
      CALL CADD(KHA2,KHE2,DQ11A2,DQ11E2,XAA,XAE)
      XAA=CML*DQ11A1
      XAE=DQ11E1
      XBA=DCMU*Q11A1
      XBE=Q11E1
      CALL CADD(XCA,XCE,XAA,XAE,XBA,XBE)
      XAA=-Q11*Q11A1
      XAE=Q11E1
      CALL CADD(DKHA1,DKHE1,XAA,XAE,XCA,XCE)
      XAA=CML*DQ11A2
      XAE=DQ11E2
      XBA=DCMU*Q11A2
      XBE=Q11E2
      CALL CADD(XCA,XCE,XAA,XAE,XBA,XBE)
      XAA=-Q11*Q11A2
      XAE=Q11E2
      CALL CADD(DKHA2,DKHE2,XAA,XAE,XCA,XCE)
      RETURN
C*****
      END
```

```
1         FUNCTION CTANH(Z)
2  C***********
3  C      THIS FUNCTION SUBROUTINE EVALUATES TANH(Z) FOR COMPLEX Z
4  C***********
5         IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
6         COMPLEX CTANH,Z,ZSQ,E,EINV,CEXP
7         DATA SMALL/0.70710687119E-1/
8         DATA D1,D3,D5,D7,D9/1.0,-0.33333333333,0.13333333333,
9        *    -0.53968253968E-1,0.21869488536E-1/
10        IF((DABS(REAL(Z)).GT.SMALL).OR.(DABS(AIMAG(Z)).GT.SMALL)) THEN
11        E=CEXP(Z)
12        EINV=1.0/E
13        CTANH=(E-EINV)/(E+EINV)
14        ELSE
15        ZSQ=Z*Z
16        CTANH=Z*(D1+ZSQ*(D3+ZSQ*(D5+ZSQ*(D7+ZSQ*D9))))
17        END IF
18        RETURN
19        END
```

```
      FUNCTION CSECH2(Z)
C**********
C     THIS FUNCTION SUBROUTINE EVALUATES SECH(Z)*SECH(Z)
C     FOR COMPLEX Z
C**********
      COMPLEX CSECH2,Z,E,EINV,CEXP
      E=CEXP(2.0D0*Z)
      EINV=1.0D0/E
      CSECH2=4.0D0/(E+2.0D0+EINV)
       RETURN
       END
```

```
      SUBROUTINE FCTVLX(Q11S,DETA,DETE)
C**********
C     THIS SUBROUTINE EVALUATES THE MODAL FUNCTION 'DET(DELTA(Q11))'
C
C     THE VALUES ARE RETURNED AS A COMPLEX AMPLITUDE 'DETA' AND A
C     REAL EXPONENT 'DETE'. THE VALUE OF THE MODAL FUNCTION IS THEN
C     'DET(DELTA(Q11))=DETA*EXP(DETE)'
C
C     SUBROUTINES USED - HNKLX,KHNKLX,CADD,KHATX
C**********
      IMPLICIT DOUBLE PRECISION*6  (A-H,O-Z)
      DOUBLE PRECISION*6 KHE1,KHE2
      COMPLEX Q11,Q12,Q22,Q23,Q33,I,SQNG,Q11S,
     $   KHA1,KHA2,DKHA1,DKHA2,Q12A1,Q12A2,DQ12A1,DQ12A2,
     $   Q22A1,Q22A2,DQ22A1,DQ22A2,Q23A1,Q23A2,DQ23A1,DQ23A2,
     $   Q33A1,Q33A2,DQ33A1,DQ33A2,TA,TB,ZETA,PSI,
     $   XKA,XKB,PKA,PKB,CHI,PHI,DETA
      COMMON/COMONE/C1,C3,C4,C5,C6,C7,C8,C9,SGNA2
      COMMON/COMTWO/ETAABS,ALPHA1,H,C11,SQNG
      DATA I/(0.0,1.0)/
C*****
C     EVALUATE Q12,Q22,Q23,Q33
C
C
C     THE FOLLOWING STATEMENT SHIFTS Q11 IN ORDER TO AVOID HAVING A
C     MESH LINE ON THE REAL AXIS. THIS SHIFT IN Q11 IS COMPENSATED
C     FOR IN THE MAIN PROGRAM.
C
      Q11=Q11S-I*0.003
C
      Q12=Q11+C4
      Q22=C5*Q11+C6
      Q23=C5*Q11+C7
      Q33=C8*Q11+C9
C
C     EVALUATE MODIFIED HANKEL FUNCTINS NEEDED
C
      CALL KHNKLX(Q12,Q12A1,Q12E1,Q12A2,Q12E2,DQ12A1,DQ12E1,
     $   DQ12A2,DQ12E2)
      CALL KHNKLX(Q22,Q22A1,Q22E1,Q22A2,Q22E2,DQ22A1,DQ22E1,DQ22A2,
     $   DQ22E2)
      CALL KHNKLX(Q23,Q23A1,Q23E1,Q23A2,Q23E2,DQ23A1,DQ23E1,
     $   DQ23A2,DQ23E2)
      CALL HNKLX(Q33,Q33A1,Q33E1,Q33A2,Q33E2,DQ33A1,DQ33E1,
     $   DQ33A2,DQ33E2)
      CALL KHATX(Q11,KHA1,KHE1,KHA2,KHE2,DKHA1,DKHE1,DKHA2,DKHE2)
C
C     EVALUATE ZETA(Q11)
C
      TA=Q12A2*KHA1
      TAE=Q12E2+KHE1
      TB=-Q12A1*KHA2
      TBE=Q12E1+KHE2
      CALL CADD(ZETA,ZETAE,TA,TAE,TB,TBE)
C
```

```
C      EVALUATE PSI(Q11)
C
       TA=DQ12A2*KHA1
       TAE=DQ12E2+KHE1
       TB=-DQ12A1*KHA2
       TBE=DQ12E1+KHE2
       CALL CADD(PSI,PSIE,TA,TAE,TB,TBE)
C
C      EVALUATE MODIFIED HANKEL FUNCTION COMBINATIONS NEEDED FOR
C      CALCULATING CHI(Q11) AND PHI(Q11)
C
       TA=DQ22A1*Q23A2
       TAE=DQ22E1+Q23E2
       TB=-DQ22A2*Q23A1
       TBE=DQ22E2+Q23E1
       CALL CADD(XKA,XKAE,TA,TAE,TB,TBE)
       TA=DQ22A1*DQ23A2
       TAE=DQ22E1+DQ23E2
       TB=-DQ22A2*DQ23A1
       TBE=DQ22E2+DQ23E1
       CALL CADD(XKB,XKBE,TA,TAE,TB,TBE)
       TA=Q22A1*Q23A2
       TAE=Q22E1+Q23E2
       TB=-Q22A2*Q23A1
       TBE=Q22E2+Q23E1
       CALL CADD(PKA,PKAE,TA,TAE,TB,TBE)
       TA=Q22A1*DQ23A2
       TAE=Q22E1+DQ23E2
       TB=-Q22A2*DQ23A1
       TBE=Q22E2+DQ23E1
       CALL CADD(PKB,PKBE,TA,TAE,TB,TBE)
C
C      EVALUATE CHI(Q11)
C
       TA=DQ33A2*XKA
       TAE=DQ33E2+XKAE
       TB=SGNA2*C3*Q33A2*XKB
       TBE=Q33E2+XKBE
       CALL CADD(CHI,CHIE,TA,TAE,TB,TBE)
C
C      EVALUATE PHI(Q11)
C
       TA=DQ33A2*PKA
       TAE=DQ33E2+PKAE
        TB=SGNA2*C3*Q33A2*PKB
        TBE=Q33E2+PKBE
        CALL CADD(PHI,PHIE,TA,TAE,TB,TBE)
C
C      EVALUATE *DET(DELTA(Q11))*
C
        TA=SGNA2*C1*ZETA*CHI
        TAE=ZETAE*CHIE
        TB=PSI*PHI
        TBE=PSIE*PHIE
        CALL CADD(DETA,DETE,TA,TAE,TB,TBE)
        DETA=SGNA2*DETA/C3
        RETURN
C*****
        END
```

```
      SUBROUTINE FDFDTX(Q11S,DETA,DETE,DETPA,DETPE)
C**********
C     THIS SUBROUTINE EVALUATES 'DET(DELTA(Q11))'
C     AND ITS DERIVATIVE WITH RESPECT TO Q11
C
C     THESE VALUES ARE RETURNED AS A COMPLEX AMPLITUDE AND A
C     REAL EXPONENT. WE HAVE
C         DET(DELTA(Q11))=DETA*EXP(DETE)
C         (DET(DELTA(Q11)))'=DETPA*EXP(DETPE)
C
C     SUBROUTINES USED - HNKLX,KHNKLX,CADD,KHATX
C**********
      IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
      DOUBLE PRECISION*6 KHE1,KHE2
      COMPLEX   Q11,Q12,Q22,Q23,Q33,I,SGNG,Q11S,
     $   KHA1,KHA2,DKHA1,DKHA2,Q12A1,Q12A2,DQ12A1,DQ12A2,
     $   Q22A1,Q22A2,DQ22A1,DQ22A2,Q23A1,Q23A2,DQ23A1,DQ23A2,
     $   Q33A1,Q33A2,DQ33A1,DQ33A2,TA,TB,TC,TD,TE,TF,TG,
     $   ZETA,PSI,DZETA,DPSI,XKA,XKB,PKA,
     $   PKB,XKAP,XKBP,XKCP,XKDP,CHI,PHI,DCHI,DPHI,DETA,DETPA
      COMMON/COMONE/C1,C3,C4,C5,C6,C7,C8,C9,SGNA2
      COMMON/COMTWO/ETAABS,ALPHA1,H,C11,SGNG
      DATA I/(0.0,1.0)/
C*****
C     EVALUATE Q12,Q22,Q23,Q33
C
C
C     THE FOLLOWING STATEMENT SHIFTS Q11 IN ORDER TO AVOID HAVING A
C     MESH LINE ON THE REAL AXIS. THIS SHIFT IN Q11 IS COMPENSATED
C     FOR IN THE MAIN PROGRAM.
C
      Q11=Q11S-I*0.003
C
      Q12=Q11+C4
      Q22=C5*Q11+C6
      Q23=C5*Q11+C7
      Q33=C8*Q11+C9
C
C     EVALUATE MODIFIED HANKEL FUNCTIONS NEEDED
C
      CALL KHNKLX(Q12,Q12A1,Q12E1,Q12A2,Q12E2,DQ12A1,DQ12E1,
     *DQ12A2,DQ12E2)
      CALL KHNKLX(Q22,Q22A1,Q22E1,Q22A2,Q22E2,DQ22A1,DQ22E1,
     $  DQ22A2,DQ22E2)
      CALL KHNKLX(Q23,Q23A1,Q23E1,Q23A2,Q23E2,DQ23A1,DQ23E1,
     $    DQ23A2,DQ23E2)
      CALL HNKLX(Q33,Q33A1,Q33E1,Q33A2,Q33E2,DQ33A1,DQ33E1,
     $    DQ33A2,DQ33E2)
      CALL KHATX(Q11,KHA1,KHE1,KHA2,KHE2,DKHA1,DKHE1,DKHA2,DKHE2)
C
C     EVALUATE ZETA(Q11)
C
      TA=Q12A2*KHA1
      TAE=Q12E2+KHE1
      TB=-Q12A1*KHA2
```

```
      TBE=Q12E1+KHE2
      CALL CADD(ZETA,ZETAE,TA,TAE,TB,TBE)
C
C     EVALUATE PSI(Q11)
C
      TA=DQ12A2*KHA1
      TAE=DQ12E2+KHE1
      TB=-DQ12A1*KHA2
      TBE=DQ12E1+KHE2
      CALL CADD(PSI,PSIE,TA,TAE,TB,TBE)
C
C     EVALUATE ZETA'(Q11)
C
      TA=DQ12A2*KHA1
      TAE=DQ12E2+KHE1
      TB=Q12A2*DKHA1
      TBE=Q12E2+DKHE1
      CALL CADD(TC,TCE,TA,TAE,TB,TBE)
      TA=-DQ12A1*KHA2
      TAE=DQ12E1+KHE2
      TB=-Q12A1*DKHA2
      TBE=Q12E1+DKHE2
      CALL CADD(TD,TDE,TA,TAE,TB,TBE)
      CALL CADD(DZETA,DZETAE,TC,TCE,TD,TDE)
C
C     EVALUATE PSI'(Q11)
C
      TA=-Q12+Q12A2*KHA1
      TAE=Q12E2+KHE1
      TB=DQ12A2*DKHA1
      TBE=DQ12E2+DKHE1
      CALL CADD(TC,TCE,TA,TAE,TB,TBE)
      TA=Q12+Q12A1*KHA2
      TAE=Q12E1+KHE2
      TB=-DQ12A1*DKHA2
      TBE=DQ12E1+DKHE2
      CALL CADD(TD,TDE,TA,TAE,TB,TBE)
      CALL CADD(DPSI,DPSIE,TC,TCE,TD,TDE)
C
C     EVALUATE MODIFIED HANKEL FUNCTION COMBINATIONS NEEDED FOR
C     CALCULATING CHI(Q11),CHI'(Q11),PHI(Q11) AND PHI'(Q11)
C
      TA=DQ22A1*Q23A2
      TAE=DQ22E1+Q23E2
       TB=-DQ22A2*Q23A1
       TBE=DQ22E2+Q23E1
       CALL CADD(XKA,XKAE,TA,TAE,TB,TBE)
       TA=DQ22A1*DQ23A2
       TAE=DQ22E1+DQ23E2
       TB=-DQ22A2*DQ23A1
       TBE=DQ22E2+DQ23E1
       CALL CADD(XKB,XKBE,TA,TAE,TB,TBE)
       TA=Q22A1*Q23A2
       TAE=Q22E1+Q23E2
       TB=-Q22A2*Q23A1
```

```
      TBE=Q22E2*Q23E1
      CALL CADD(PKA,PKAE,TA,TAE,TB,TBE)
      TA=Q22A1*DQ23A2
      TAE=Q22E1*DQ23E2
      TB=-Q22A2*DQ23A1
      TBE=Q22E2*DQ23E1
      CALL CADD(PKB,PKBE,TA,TAE,TB,TBE)
      TA=Q22A2*Q23A1
      TAE=Q22E2*Q23E1
      TB=-Q22A1*Q23A2
      TBE=Q22E1*Q23E2
      CALL CADD(XKAP,XKAPE,TA,TAE,TB,TBE)
      TA=DQ22A1*DQ23A2
      TAE=DQ22E1*DQ23E2
      TB=-DQ22A2*DQ23A1
      TBE=DQ22E2*DQ23E1
      CALL CADD(XKBP,XKBPE,TA,TAE,TB,TBE)
      TA=Q22A2*DQ23A1
      TAE=Q22E2*DQ23E1
      TB=-Q22A1*DQ23A2
      TBE=Q22E1*DQ23E2
      CALL CADD(XKCP,XKCPE,TA,TAE,TB,TBE)
      TA=DQ22A2*Q23A1
      TAE=DQ22E2*Q23E1
      TB=-DQ22A1*Q23A2
      TBE=DQ22E1*Q23E2
      CALL CADD(XKDP,XKDPE,TA,TAE,TB,TBE)
C
C     EVALUATE CHI(Q11)
C
      TA=DQ33A2*XKA
      TAE=DQ33E2*XKAE
      TB=SGNA2*C3*Q33A2*XKB
      TBE=Q33E2*XKBE
      CALL CADD(CHI,CHIE,TA,TAE,TB,TBE)
C
C     EVALUATE PHI(Q11)
C
      TA=DQ33A2*PKA
      TAE=DQ33E2*PKAE
      TB=SGNA2*C3*Q33A2*PKB
      TBE=Q33E2*PKBE
      CALL CADD(PHI,PHIE,TA,TAE,TB,TBE)
C
C     EVALUATE CHI*(Q11)
C
      TA=-Q33*C8*Q33A2*XKA
      TAE=Q33E2*XKAE
      TB=C5*DQ33A2*Q22*XKAP
      TBE=DQ33E2*XKAPE
      TC=C5*DQ33A2*XKBP
      TCE=DQ33E2*XKBPE
      TD=SGNA2*C3*C8*DQ33A2*XKB
      TDE=DQ33E2*XKBE
      TE=SGNA2*C3*C5*Q33A2*Q22*XKCP
```

```
      TEE=Q33E2*XKCPE
      TF=SGNA2*C3*C5*Q33A2*Q23*XKDP
      TFE=Q33E2*XKDPE
      CALL CADD(TG,TGE,TA,TAE,TB,TBE)
      CALL CADD(TA,TAE,TG,TGE,TC,TCE)
      CALL CADD(TB,TBE,TA,TAE,TD,TDE)
      CALL CADD(TA,TAE,TB,TBE,TE,TEE)
      CALL CADD(DCHI,DCHIE,TA,TAE,TF,TFE)
C
C     EVALUATE PHI'(Q11)
C
      TA=-Q33*C8*Q33A2*PKA
      TAE=Q33E2*PKAE
      TB=C5*DQ33A2*XKA
      TBE=DQ33E2*XKAE
      CALL CADD(TC,TCE,TA,TAE,TB,TBE)
      TA=C5*DQ33A2*PKB
      TAE=DQ33E2*PKBE
      CALL CADD(TB,TBE,TA,TAE,TC,TCE)
      TA=SGNA2*C3*C8*DQ33A2*PKB
      TAE=DQ33E2*PKBE
      CALL CADD(TC,TCE,TB,TBE,TA,TAE)
      TA=SGNA2*C3*C5*Q33A2*XKB
      TAE=Q33E2*XKBE
      CALL CADD(TB,TBE,TC,TCE,TA,TAE)
      TA=-SGNA2*C3*C5*Q33A2*Q23*PKA
      TAE=Q33E2*PKAE
      CALL CADD(DPHI,DPHIE,TB,TBE,TA,TAE)
C
C     EVALUATE DET(DELTA(Q11))
C
      TA=SGNA2*C1*ZETA*CHI
      TAE=ZETAE*CHIE
      TB=PSI*PHI
      TBE=PSIE*PHIE
      CALL CADD(DETA,DETE,TA,TAE,TB,TBE)
      DETA=SGNA2*DETA/C3
C
C     EVALUATE (DET(DELTA(Q11)))'
C
      TA=SGNA2*C1*DZETA*CHI
      TAE=DZETAE*CHIE
      TB=SGNA2*C1*ZETA*DCHI
      TBE=ZETAE*DCHIE
      CALL CADD(TC,TCE,TA,TAE,TB,TBE)
      TA=DPSI*PHI
      TAE=DPSIE*PHIE
      TB=PSI*DPHI
      TBE=PSIE*DPHIE
      CALL CADD(TD,TDE,TA,TAE,TB,TBE)
      CALL CADD(DETPA,DETPE,TC,TCE,TD,TDE)
      DETPA=SGNA2*DETPA/C3
      RETURN
C*****
      END
```

```
      SUBROUTINE FZEROX(TLEFT,TRIGHT,TBOT,TTOP,TMSH0,TOL,MPRINT,ZEROS,
     $                  NRZ)
C  FOR ROOT-FINDER
C  THIS VERSION UPDATED TO MODIFY GMAX AND TMESH.  JAN 6 1982.
C  FZEROS IS A ROUTINE FOR FINDING THE ZEROS OF A COMPLEX FUNCTION, F,
C                WHICH LIE WITHIN A SPECIFIED RECTANGULAR REGION OF THE
C                COMPLEX (THETA) PLANE, PROVIDED THE FUNCTION HAS NO
C                POLES IN THE VICINITY OF THE RECTANGLE.
C  EXPLANATION OF PARAMETERS--
C         TLEFT - VALUE OF REAL PART OF THETA AT LEFT EDGE OF RECTANGLE.
C         TRIGHT - VALUE OF REAL PART OF THETA AT RT EDGE OF RECTANGLE.
C         TBOT - VALUE OF IMAG PART OF THETA AT BOTTOM EDGE OF RECTANGLE.
C         TTOP - VALUE OF IMAG PART OF THETA AT TOP EDGE OF RECTANGLE.
C         TMESH - SET EQUAL TO ABOUT HALF THE AVERAGE SPACING BETWEEN
C                 ZEROS WITHIN THE RECTANGLE.  A SMALLER VALUE MAY BE USED
C                 AS A SAFETY MEASURE, BUT TOO SMALL A VALUE WILL RESULT
C                 IN EXCESSIVE RUN TIME.
C         TOL - TOLERANCE TO WHICH ZEROS ARE TO BE FOUND.  IF TWO
C                 ZEROS ARE CLOSER THAN 'TOL', THE ROOT-FINDER WILL STOP
C                 WITH AN ERROR MESSAGE.
C         MPRINT - NORMALLY SET TO ZERO.  A NON-ZERO VALUE LEADS TO
C                 PRINT-OUT FOR DEBUGGING.
C         ZEROS - OUTPUT LIST OF (COMPLEX) VALUES OF THETA AT WHICH
C                 ZEROS ARE FOUND.
C         NR Z - THE NUMBER OF ZEROS FOUND.
C  SUBROUTINES TO BE PROVIDED--
C     FCTVLX(Z,F,FE) - TO RETURN THE VALUE OF THE FUNCTION, F*EXP(FE),
C        AT THE POINT IN THE COMPLEX PLANE SPECIFIED BY Z.
C     FDFDTX(Z,F,FE,DFDT,DFDTE) - SAME AS 'FCTVLX' EXCEPT THAT THE
C        DERIVATIVE, DFDT*EXP(DFDTE), OF THE FUNCTION WRT Z MUST ALSO
C        BE RETURNED.
C
      IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
      COMMON/NEWMSH/NEWMSH
      COMMON/TMC COM/TMESH
C
      DIMENSION ZEROS(2,1)
      COMPLEX F,PREV F,F00,F10,F01,F11,PARTS
      DIMENSION PART(2),SOL(2),THETA(2)
      DIMENSION K EDGE1(50),K EDGE2(50),K EDGE3(50),K EDGE4(50)
      EQUIVALENCE (PART(1),PARTS)
      MAXNS0=MAX0(INT(4.0*(TTOP-TBOT)/TMSH0),INT(4.0*(TRIGHT-TLEFT)
     $   /TMSH0))
      MAXNT=2
C
C
      IF (MPRINT.NE.0) WRITE(11,906)
  906 FORMAT ('1')
C
      TMESH = TMSH0
      NTIME = 0
      GO TO 7
    5 TMESH = TMESH/2.0
      NTIME = NTIME+1
      IF (NTIME .GT. MAX NT) GO TO 97
```

```
    7    NEWMSH = 0
C
C  SIDES OF RECTANGLE IN TMESH UNITS
      JLT=INT(TLEFT/TMESH)
      IF(TLEFT.GT.0.0) JLT=JLT-1
      IF (TLEFT .LT. 0.0) JLT = JLT-2
      IF(TLEFT.EQ.0.0) JLT=0
C
      JRT=INT(TRIGHT/TMESH)
      IF (TRIGHT .GT. 0.0) JRT = JRT+2
      IF(TRIGHT.LT.0.0) JRT=JRT+1
      IF(TRIGHT.EQ.0.0) JRT=0
C
      JBOT = INT(TBOT/TMESH)
      IF (TBOT .GE. 0.0) JBOT = JBOT-1
      IF (TBOT .LT. 0.0) JBOT = JBOT-2
C
      JTOP = INT(TTOP/TMESH)
      IF (TTOP .GT. 0.0) JTOP = JTOP+2
      IF (TTOP .LE. 0.0) JTOP = JTOP+1
C
C  INITIALIZATION OF PARAMETERS.
      KI = JTOP
      KR = JLT
      KEDGE = 1
      CALL FINDFX(KR,KI,F,FE,TLEFT,TRIGHT)
      E00=1.0
      E01=1.0
      E10=1.0
      E11=1.0
      NR E1 = 0
      NR E2 = 0
      NR E3 = 0
      NR E4 = 0
      NR Z = 0
      GO TO 15
C
   10 IF(NR ZL .LE. 1) GO TO 15
      WRITE(11,2000) NRZL
 2000 FORMAT (/0/,I3,1X,/MODES FOUND ON SAME PHASE LINE/)
      GO TO 5
   15 NR ZL = 0
      NRSOU = 0
C
   20 PREVF=F
      PREVFE=FE
      IF(MPRINT.NE.0) WRITE(11,2200) KR,KI,F,FE
 2200 FORMAT(30X,2I4,5X,'F = ',1P2E12.3,5X,'FE = ',1PE12.3)
      GO TO (21,26,31,36),KEDGE
C
C  SEARCH ALONG LEFT EDGE OF RECTANGLE FOR SIGN CHANGES IN IMAG(F).
   21 IF (KI .EQ. JBOT) KEDGE = 2
      IF (KEDGE .EQ. 2) GO TO 26
      KI = KI-1
      CALL FINDFX(KR,KI,F,FE,TLEFT,TRIGHT)
```

```
      IF ((AIMAG(PREV F) .GT. 0.0 .AND. AIMAG(F) .GT. 0.0)
     $           .OR. (AIMAG(PREV F) .LT. 0.0 .AND. AIMAG(F) .LT. 0.0))
     $           GO TO 20
      IF (NR E1 .EQ. 0) GO TO 23
      DO 22 K=1,NR E1
      IF (KI .EQ. K EDGE1(K)) GO TO 20
   22 CONTINUE
   23 F01 = PREV F
      F01E=PREVFE
      F00 = F
      F00E=FE
      LI = KI
      LR = JLT
      GO TO 43
C
C  SEARCH ALONG BOTTOM EDGE OF RECTANGLE FOR SIGN CHANGES IN IMAG(F).
   26 IF (KR .EQ. JRT) KEDGE = 3
      IF (KEDGE .EQ. 3) GO TO 31
      KR = KR+1
      CALL FINDFX(KR,KI,F,FE,TLEFT,TRIGHT)
      IF ((AIMAG(PREV F) .GT. 0.0 .AND. AIMAG(F) .GT. 0.0)
     $           .OR. (AIMAG(PREV F) .LT. 0.0 .AND. AIMAG(F) .LT. 0.0))
     $           GO TO 20
      IF (NR E2 .EQ. 0) GO TO 28
      DO 27 K=1,NR E2
      IF (KR .EQ. K EDGE2(K)) GO TO 20
   27 CONTINUE
   28 F00 = PREV F
      F00E=PREVFE
      F10 = F
      F10E=FE
      LI = JBOT
      LR = KR-1
      GO TO 48
C
C  SEARCH ALONG RIGHT EDGE OF RECTANGLE FOR SIGN CHANGES IN IMAG(F).
   31 IF (KI .EQ. JTOP) KEDGE = 4
      IF (KEDGE .EQ. 4) GO TO 36
      KI = KI+1
      CALL FINDFX(KR,KI,F,FE,TLEFT,TRIGHT)
      IF ((AIMAG(PREV F) .GT. 0.0 .AND. AIMAG(F) .GT. 0.0)
     $           .OR. (AIMAG(PREV F) .LT. 0.0 .AND. AIMAG(F) .LT. 0.0))
     $           GO TO 20
      IF (NR E3 .EQ. 0) GO TO 33
      DO 32 K=1,NR E3
      IF (KI .EQ. K EDGE3(K)) GO TO 20
   32 CONTINUE
   33 F10 = PREV F
      F10E=PREVFE
      F11 = F
      F11E=FE
      LI = KI-1
      LR = JRT-1
      GO TO 53
C
```

```
C  SEARCH ALONG TOP EDGE OF RECTANGLE FOR SIGN CHANGES IN IMAG(F).
   36 IF (KR .EQ. JLT) GO TO 80
      KR = KR-1
      CALL FINDFX(KR,KI,F,FE,TLEFT,TRIGHT)
      IF ((AIMAG(PREV F) .GT. 0.0 .AND. AIMAG(F) .GT. 0.0)
     $           .OR. (AIMAG(PREV F) .LT. 0.0 .AND. AIMAG(F) .LT. 0.0))
     $           GO TO 20
      IF (NR E4 .EQ. 0) GO TO 38
      DO 37 K=1,NR E4
      IF (KR .EQ. K EDGE4(K)) GO TO 20
   37 CONTINUE
   38 F11 = PREV F
      F11F=PREVFE
      F01 = F
      F01F=FE
      LI = JTOP-1
      LR = KR
      GO TO 58
C
C  ENTER MESH SQUARE FROM LEFT SIDE OR EXIT RECTANGLE AT RIGHT EDGE.
   41 LR = LR+1
      IF (LR .LE. JRT-1) GO TO 42
      NR E3 = NR E3+1
      K EDGE3(NR E3) = LI+1
      IF(MPRINT.NE.0) WRITE(11,990)
      GO TO 10
   42 F01=F11/E11
      F01E=F11E
      F00=F10/E10
      F00E=F10E
   43 CALL FINDFX(LR+1,LI+1,F11,F11E,TLEFT,TRIGHT)
      CALL FINDFX(LR+1,LI,F10,F10E,TLEFT,TRIGHT)
      SCALE=AMAX1(F00E,F01E,F10E,F11E)
      IF(SCALE.GT.34.0) THEN
      E00=EXP(F00E-SCALE)
      E01=EXP(F01E-SCALE)
      E10=EXP(F10E-SCALE)
      E11=EXP(F11E-SCALE)
      F00=F00*E00
      F01=F01*E01
      F10=F10*E10
      F11=F11*E11
      END IF
      LEDGE = 1
      ENTER R = 0.0
      ENTER I = -AIMAG(F00)/AIMAG(F01-F00)
      GO TO 60
C
C  ENTER MESH SQUARE FROM BOTTOM SIDE OR EXIT RECTANGLE AT TOP EDGE.
   46 LI = LI+1
      IF (LI .LE. JTOP-1) GO TO 47
      NR E4 = NR E4+1
      K EDGE4(NR E4) = LR
      IF(MPRINT.NE.0) WRITE(11,990)
      GO TO 10
```

```
221     47 F00=F01/E01
222        F00F=F01F
223        F10=F11/E11
224        F10F=F11F
225     48 CALL FINDFX(LR,LI+1,F01,F01E,TLEFT,TRIGHT)
226        CALL FINDFX(LR+1,LI+1,F11,F11E,TLEFT,TRIGHT)
227        SCALE=AMAX1(F00E,F01E,F10F,F11E)
228        IF(SCALE.GT.34.0) THEN
229        E00=EXP(F00E-SCALE)
230        F01=EXP(F01E-SCALE)
231        F10=EXP(F10E-SCALE)
232        E11=EXP(F11E-SCALE)
233        F00=F00*E00
234        F01=F01*E01
235        F10=F10*E10
236        F11=F11*E11
237        END IF
238        LEDGE = 2
239        ENTER R = -AIMAG(F00)/AIMAG(F10-F00)
240        ENTER I = 0.0
241        GO TO 60
242  C
243  C  ENTER MESH SQUARE FROM RIGHT SIDE OR EXIT RECTANGLE AT LEFT EDGE.
244     51 LR = LR-1
245        IF (LR .GE. JLT) GO TO 52
246        NR E1 = NR E1+1
247        K EDGE1(NR E1) = LI
248        IF(MPRINT.NE.0)  WRITE(11,990)
249        GO TO 10
250     52 F11=F01/E01
251        F11E=F01E
252        F10=F00/E00
253        F10E=F00E
254     53 CALL FINDFX(LR,LI+1,F01,F01E,TLEFT,TRIGHT)
255        CALL FINDFX(LR,LI,F00,F00E,TLEFT,TRIGHT)
256        SCALE=AMAX1(F00E,F01E,F10E,F11E)
257        IF(SCALE.GT.34.0) THEN
258        E00=EXP(F00E-SCALE)
259        E01=EXP(F01E-SCALE)
260        E10=EXP(F10E-SCALE)
261        E11=EXP(F11E-SCALE)
262        F00=F00*E00
263        F01=F01*E01
264        F10=F10*E10
265        F11=F11*E11
266        END IF
267        LEDGE = 3
268        ENTER R = 1.0
269        ENTER I = -AIMAG(F10)/AIMAG(F11-F10)
270        GO TO 60
271  C
272  C  ENTER MESH SQUARE FROM TOP SIDE OR EXIT RECTANGLE AT BOTTOM EDGE.
273     56 LI = LI+1
274        IF (LI .GE. JROT) GO TO 57
275        NR E2 = NR E2+1
```

```
      K EDGE2(NR E2) = LR+1
      IF(MPRINT.NE.0) WRITE(11,990)
      GO TO 10
   57 F01=F00/E00
      F01F=F00E
      F11=F10/E10
      F11E=F10E
   58 CALL FINDFX(LR,LI,F00,F00E,TLEFT,TRIGHT)
      CALL FINDFX(LR+1,LI,F10,F10E,TLEFT,TRIGHT)
      SCALE=AMAX1(F00E,F01E,F10E,F11E)
      IF(SCALE.GT.34.0) THEN
      E00=EXP(F00E-SCALE)
      E01=EXP(F01E-SCALE)
      E10=EXP(F10E-SCALE)
      E11=EXP(F11E-SCALE)
      F00=F00*E00
      F01=F01*E01
      F10=F10*E10
      F11=F11*E11
      END IF
      LEDGE = 4
      ENTER R = -AIMAG(F01)/AIMAG(F11-F01)
      ENTER I = 1.0
      GO TO 60
C
C  FOR DEBUGGING ONLY, PRINT CO-ORDINATE OF LOWER LEFT CORNER
C                OF CURRENT MESH SQUARE.  RESULTING SET OF PRINTED
C                CO-ORDINATES GIVES TRACE OF EACH LINE ALONG WHICH
C                IMAG(F)=0.0 FROM ITS ENTRY ON THE EDGE OF THE
C                RECTANGLE TO ITS EXIT AT ANOTHER POINT ON THE
C                RECTANGLE.
   60 IF(MPRINT.NE.0) WRITE(11,960) LR,LI
  960 FORMAT (' ',20X,2I5)
C  TEST FOR THERE BEING TWO (HYPERBOLIC) LINES ENTERING AND LEAVING
C                THE MESH SQUARE ALONG EACH OF WHICH IMAG(F)=0.0
C                IF SO, SET 'LTWO' NON-ZERO.
      LTWO = 0
      NRSQU=NRSQU+1
      IF(NRSQU .GT. MAXNSQ) GO TO 95
      IF(((AIMAG(F00).GE. 0.0).AND.(AIMAG(F11).GE. 0.0)
     $   .AND.(AIMAG(F10).LE. 0.0).AND.(AIMAG(F01).LE. 0.0))
     $   .OR.((AIMAG(F00).LE. 0.0).AND.(AIMAG(F11).LE. 0.0)
     $   .AND.(AIMAG(F10).GE. 0.0).AND.(AIMAG(F01).GE. 0.0)))  LTWO=1
C  TEST FOR THERE BEING AT LEAST ONE (HYPERBOLIC) LINE ENTERING AND
C                LEAVING THE MESH SQUARE ALONG WHICH REAL(F)=0.0
C                IF NOT, SET 'I90' TO ZERO.
      I90 = 1
      IF ((REAL(F00) .GT. 0.0 .AND. REAL(F10) .GT. 0.0
     $          .AND. REAL(F01) .GT. 0.0 .AND. REAL(F11) .GT. 0.0)
     $          .OR. (REAL(F00) .LT. 0.0 .AND. REAL(F10) .LT. 0.0
     $          .AND. REAL(F01) .LT. 0.0 .AND. REAL(F11) .LT. 0.0))
     $          I90 = 0
      IF (LTWO .EQ. 0 .AND. I90 .EQ. 0) GO TO 70
C
C  COMPUTATION OF COEFFICIENTS TO BE USED IN DESCRIBING THE VARIATION
```

```
C                OF THE   FUNCTION WITHIN A MESH SQUARE GIVEN THE
C                VALUES AT THE CORNERS OF THE SQUARE AND LINEAR
C                VARIATION ALONG ITS EDGES.  ALSO, THE POSITION OF
C                CROSSING OF THE HYPERBOLIC ASYMPTOTES (WHICH ARE
C                PARALLEL TO  THE SIDES OF THE SQUARE) FOR THE LINES
C                IMAG(F)=0.0 ARE COMPUTED IF BOTH LINES (BRANCHES)
C                ENTER AND LEAVE  THE SQUARE.
      PARTS = F00
      AR = PART(1)
      AI = PART(2)
      PARTS = F01-F00
      BR = PART(1)
      BI = PART(2)
      PARTS = F10-F00
      CR = PART(1)
      CI = PART(2)
      PARTS = F00+F11-F01-F10
      DR = PART(1)
      DI = PART(2)
C
      IF (LTWO .EQ. 0) GO TO 61
      CENTR R = -BI/DI
      CENTR I = -CI/DI
      IF (I90 .EQ. 0) GO TO 70
C
   61 AB = AR*BI-AI*BR
      AC = AR*CI-AI*CR
      AD = AR*DI-AI*DR
      BC = BR*CI-BI*CR
      BD = BR*DI-BI*DR
      CD = CR*DI-CI*DR
C
      IF (ABS(BD) .LT. ABS(CD)) GO TO 64
C  SOLUTION FOR TWO POSSIBLE POINTS AT WHICH THERE MAY BE CROSSINGS
C                OF THE (HYPERBOLIC) LINES REAL(F)=0.0 AND IMAG(F)=0.0.
C                A CROSSING POINT IS CHOSEN TO BE A ZERO OF THE
C                  FUNCTION IF IT LIES WITHIN THE CURRENT MESH SQUARE
C                AND IF IT LIES ON THE BRANCH OF IMAG(F)=0.0 CURRENTLY
C                BEING FOLLOWED.  MULTIPLE CROSSINGS ALONG THIS BRANCH
C                (ACTUALLY A SERIES OF CONNECTING HYPERBOLIC BRANCHES)
C                IN THE SAME OR IN DIFFERENT SQUARES ARE NOTED SINCE
C                THESE MUST LATER BE RESOLVED.
      A = CD
      B = (AD-BC)*0.5
      C = AB
      CALL QUAD (A,B,C,SOL,NR SOL)
      IF (NR SOL .EQ. 0) GO TO 70
      DO 63 N=1,NR SOL
      UREAL = SOL(N)
      IF (UREAL .LT. 0.0 .OR. UREAL .GT. 1.0) GO TO 63
      D1 = BR+DR*UREAL
      D2 = BI+DI*UREAL
      IF (ABS(D1) .GE. ABS(D2)) UIMAG = -(AR+CR*UREAL)/D1
      IF (ABS(D2) .GE. ABS(D1)) UIMAG = -(AI+CI*UREAL)/D2
      IF (UIMAG .LT. 0.0 .OR. UIMAG .GT. 1.0) GO TO 63
```

```
      IF (LTWO .EQ. 0) GO TO 62
      IF ((ENTER R-CENTR R)*(U REAL-CENTR R) .LT. 0.0
     $           .OR. (ENTER I-CENTR I)*(U IMAG-CENTR I) .LT. 0.0)
     $           GO TO 63
   62 THETA(1) = (LR+U REAL)*TMESH
      THETA(2) = (LI+U IMAG)*TMESH
      IF(MPRINT.NE.0) WRITE(11,962) LR,LI,THETA(1),THETA(2)
  962 FORMAT (' ',2I4,3X,'THETA =',2F9.4)
      NRZ = NRZ+1
      ZEROS(1,NRZ) = THETA(1)
      ZEROS(2,NRZ) = THETA(2)
      NR ZL = NR ZL+1
   63 CONTINUE
      GO TO 67
C
C  ALTERNATE SOLUTION FOR THE ABOVE.  TWO FORMS ARE NEEDED SINCE,
C                IN ANY GIVEN CASE, EITHER FORM (BUT NOT BOTH FORMS)
C                MAY BE INDETERMINATE.
   64 A = BD
      B = (AD+BC)*0.5
      C = AC
      CALL QUAD (A,B,C,SOL,NR SOL)
      IF (NR SOL .EQ. 0) GO TO 70
      DO 66 N=1,NR SOL
      UIMAG = SOL(N)
      IF (UIMAG .LT. 0.0 .OR. UIMAG .GT. 1.0) GO TO 66
      D1 = CR+DR*UIMAG
      D2 = CI+DI*UIMAG
      IF (ABS(D1) .GE. ABS(D2)) UREAL = -(AR+BR*UIMAG)/D1
      IF (ABS(D2) .GE. ABS(D1)) UREAL = -(AI+BI*UIMAG)/D2
      IF (UREAL .LT. 0.0 .OR. UREAL .GT. 1.0) GO TO 66
      IF (LTWO .EQ. 0) GO TO 65
      IF ((ENTER R-CENTR R)*(U REAL-CENTR R) .LT. 0.0
     $           .OR. (ENTER I-CENTR I)*(U IMAG-CENTR I) .LT. 0.0)
     $           GO TO 66
   65 THETA(1) = (LR+U REAL)*TMESH
      THETA(2) = (LI+U IMAG)*TMESH
      IF(MPRINT.NE.0) WRITE(11,965) LR,LI,THETA(1),THETA(2)
  965 FORMAT (' ',2I4,3X,'THETA =',2F9.4)
      NRZ = NRZ+1
      ZEROS(1,NRZ) = THETA(1)
      ZEROS(2,NRZ) = THETA(2)
      NR ZL = NR ZL+1
   66 CONTINUE
   67 CONTINUE
C
   70 CONTINUE
C  TEST FOR EXIT FROM LEFT EDGE OF MESH SQUARE.
      IF (LEDGE .EQ. 1) GO TO 72
      IF ((AIMAG(F00) .GT. 0.0 .AND. AIMAG(F01) .GT. 0.0)
     $           .OR. (AIMAG(F00) .LT. 0.0 .AND. AIMAG(F01) .LT. 0.0))
     $           GO TO 72
      IF (LTWO .EQ. 0) GO TO 51
      EXIT R = 0.0
      EXIT I = -AIMAG(F00)/AIMAG(F01-F00)
```

```
441         IF ((ENTER R-CENTR R)*(EXIT R-CENTR R) .LT. 0.0
442        $           .OR. (ENTER I-CENTR I)*(EXIT I-CENTR I) .LT. 0.0)
443        $           GO TO 72
444         GO TO 51
445 C
446 C   TEST FOR EXIT FROM BOTTOM EDGE OF MESH SQUARE.
447     72  IF (LEDGE .EQ. 2) GO TO 73
448         IF ((AIMAG(F00) .GT. 0.0 .AND. AIMAG(F10) .GT. 0.0)
449        $           .OR. (AIMAG(F00) .LT. 0.0 .AND. AIMAG(F10) .LT. 0.0))
450        $           GO TO 73
451         IF (LTWO .EQ. 0) GO TO 56
452         EXIT R = -AIMAG(F00)/AIMAG(F10-F00)
453         EXIT I = 0.0
454         IF ((ENTER R-CENTR R)*(EXIT R-CENTR R) .LT. 0.0
455        $           .OR. (ENTER I-CENTR I)*(EXIT I-CENTR I) .LT. 0.0)
456        $           GO TO 73
457         GO TO 56
458 C
459 C   TEST FOR EXIT FROM RIGHT EDGE OF MESH SQUARE.
460     73  IF (LEDGE .EQ. 3) GO TO 74
461         IF ((AIMAG(F10) .GT. 0.0 .AND. AIMAG(F11) .GT. 0.0)
462        $           .OR. (AIMAG(F10) .LT. 0.0 .AND. AIMAG(F11) .LT. 0.0))
463        $           GO TO 74
464         IF (LTWO .EQ. 0) GO TO 41
465         EXIT R = 1.0
466         EXIT I = -AIMAG(F10)/AIMAG(F11-F10)
467         IF ((ENTER R-CENTR R)*(EXIT R-CENTR R) .LT. 0.0
468        $           .OR. (ENTER I-CENTR I)*(EXIT I-CENTR I) .LT. 0.0)
469        $           GO TO 74
470         GO TO 41
471 C
472 C   TEST FOR EXIT FROM TOP EDGE OF MESH SQUARE.
473     74  IF (LEDGE .EQ. 4) GO TO 90
474         IF ((AIMAG(F01) .GT. 0.0 .AND. AIMAG(F11) .GT. 0.0)
475        $           .OR. (AIMAG(F01) .LT. 0.0 .AND. AIMAG(F11) .LT. 0.0))
476        $           GO TO 90
477         IF (LTWO .EQ. 0) GO TO 46
478         EXIT R = -AIMAG(F01)/AIMAG(F11-F01)
479         EXIT I = 1.0
480         IF ((ENTER R-CENTR R)*(EXIT R-CENTR R) .LT. 0.0
481        $           .OR. (ENTER I-CENTR I)*(EXIT I-CENTR I) .LT. 0.0)
482        $           GO TO 90
483         GO TO 46
484 C
485     80  IF(NRZ.NE.0) CALL NOMSHX(TMESH,TOL,MPRINT,ZEROS,NRZ)
486         IF (NEWMSH .NE. 0) GO TO 5
487         RETURN
488 C
489     90  WRITE(11,909) LR,LI
490    909  FORMAT ('0','NO EXIT FROM MESH SQUARE',2I6)
491         GO TO 5
492 C
493     95  WRITE(11,9500)
494   9500  FORMAT('0','TOO MANY SQUARES ON SAME PHASE LINE--MAKE',
495        $ ' TMESH SMALLER.')
```

```
      GO TO 5
C
  990 FORMAT('0')
C
   97 WRITE(11,9700)
 9700 FORMAT (' 0',' TMESH HAS BEEN REDUCED BUT PROBLEMS REMAIN IN'
     $,' EXECUTING FZEROS')
      NEWMSH = 1
      NR Z=0
      RETURN
C
      END
```

```
      SUBROUTINE QUAD (A,B,C,SOL,NR SOL)
C  FOR ROOT-FINDER
C
C  SOLUTION FOR THE REAL ROOTS OF A QUADRATIC EQUATION OF THE
C                FORM A*X**2+2.0*B*X+C=0.0, WHERE X IS CALLED 'SOL'
C                IN THIS ROUTINE.  THE NUMBER OF REAL ROOTS FOUND IS
C                GIVEN BY 'NR SOL'.  A VALUE OF 1 FOR 'NR SOL' RESULTS
C                FROM THE QUADRATIC EQUATION APPROACHING LINEARITY.
C                USED BY SUBROUTINES FZEROS AND NO MESH.
C
      IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
      DIMENSION SOL(2)
C
C
      ACOBSQ=(A/B)*(C/B)
      IF(ABS(ACOBSQ).LT.(0.3)) GO TO 20
      ARG=1.0-ACOBSQ
      NRSOL=0
      IF(ARG.LT.(0.0)) RETURN
      NRSOL=2
      ROOT=ABS(B)*SQRT(ARG)
C
C***
C
      SOL(1) = (-B+ROOT)/A
      SOL(2) = (-B-ROOT)/A
      RETURN
C
   20 TERM = 1.0
      SUM = 1.0
      DO 21 K=1,50
      TERM = TERM*((K-0.5)/(K+1.0))*ACOBSQ
      SUM = SUM+TERM
      IF (ABS(TERM) .LT. 1.0E-10) GO TO 22
   21 CONTINUE
   22 SOL(1) = -C/(2.0*B)*SUM
      NR SOL = 1
      IF (ABS(A/(2.0*B)) .LT. 1.0E-30) RETURN
      NR SOL = 2
      SOL(2) = -2.0*B/A-SOL(1)
      RETURN
C
      END
```

```
      SUBROUTINE NOMSHX(TMESH,TOL,MPRINT,ZEROS,NRZ)
C  FOR ROOT-FINDER
C  THIS VERSION UPDATED TO MODIFY GMAX AND TMESH.  JAN 6 1982.
C  ROUTINE FOR FINDING EXACT (IN THE SENSE OF NO MESH APPROXIMATION)
C                LOCATIONS OF ZEROS OF THE FUNCTION, F, FOR WHICH
C                A COMPLETE, BUT APPROXIMATE, SET WAS FOUND IN
C                SUBROUTINE FZEROS.
C                TWO DEFICIENCIES REMAIN IN THIS ROUTINE.  ONE IS
C                THAT ZEROS CLOSER THAN THE FINITE VALUE 'TOL' CANNOT
C                 BE RESOLVED.  THIS HAS CAUSED A PROBLEM A FEW TIMES
C                 IN USAGE TO DATE, BUT HAS ALWAYS BEEN RESOLVED BY
C                 USING A SMALLER VALUE OF 'TOL'.  THE SECOND PROBLEM
C                 IS THAT THE RESOLUTION OF 'MULTIPLE CROSSINGS',
C                 WHICH OCCUR BECAUSE OF THE NON-ANALITICITY OF THE
C                 MESH APPROXIMATION, IS SUCH THAT IN PRINCIPLE A
C                 ZERO CAN BE MISSED.  THERE IS NO EVIDENCE THIS HAS
C                 EVER OCCURRED IN PRACTICE.  IT MAY OR MAY NOT BE
C                 COST EFFECTIVE TO DEVELOP THE PROGRAM FURTHER TO
C                 CORRECT THESE TWO PROBLEMS.
C
       IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
       COMMON/NEWMSH/NEWMSH
       DIMENSION PART(2)
       COMPLEX ZEROS(1),
      $           THETA,
      $           F,DFDT,DELT,
      $           PARTS,
      $           ZEROS0(100)
       EQUIVALENCE (PART(1),PARTS)
C
C
       IF (MPRINT.NE.0) WRITE(11,900)
  900 FORMAT ('1','ITERATIONS TO REMOVE MESH')
C
C  NEWTON-RAPHSON (FIRST-ORDER) ITERATION TO FIND EXACT LOCATIONS
C                OF THE ZEROS OF F.
C
       DO 29 JZ=1,NRZ
       ZEROS0(JZ) = ZEROS(JZ)
       IF (MPRINT.NE.0) WRITE(11,901) JZ
  901 FORMAT (' ',I8)
   21 IF (MPRINT.NE.0) WRITE(11,902) ZEROS(JZ)
  902 FORMAT (' ',10X,2(F9.4))
       THETA = ZEROS(JZ)
       CALL FDFDTX(THETA,F,FE,DFDT,DFDTE)
       DELT=-(F/DFDT)*EXP(FE-DFDTE)
       ZEROS(JZ) = ZEROS(JZ)+DELT
       IF (CABS(DELT) .GT. 0.3*TOL) GO TO 21
   29 CONTINUE
C
C  NUMBER OF UNIQUE ZEROS COMPARED TO TOTAL NUMBER
   30 IF(NRZ   .LE. 1) RETURN
       MATCH = 0
       DO 33 J=2,NRZ
       JM1 = J-1
```

```
      DO 32 JJ=1,JM1
      IF (CABS(ZEROS(J)-ZEROS(JJ)) .GT. TOL) GO TO 32
      IF (MPRINT.NE.0) WRITE(11,905) J,JJ
  905 FORMAT ('0','ZEROS NR',I3,1X,'AND',I3,1X,
     $          'ARE THE SAME')
      MATCH = 1
   32 CONTINUE
   33 CONTINUE
C
      IF (MATCH .EQ. 0) RETURN
   40 NEWMSH = NEWMSH+1
      IF (NEWMSH.LE.2) WRITE(11,400)
  400 FORMAT ('0',' PROBLEMS IN THIS RANGER-RANGEI BOX.  PROGRAM WILL US
     $E TMESH/2 AND RE-EXECUTE THE PROGRAM')
      IF (NEWMSH.GE.3) WRITE(11,405)
  405 FORMAT ('0',' PROBLEMS IN THIS RANGER-RANGEI BOX.  USER MUST DECRE
     $ASE GMAX AND RE-EXECUTE THE PROGRAM')
      NR 7 = 0
C
      RETURN
      END
```

```
      SUBROUTINE FINDFX(JR,JI,F,FE,TLEFT,TRIGHT)
C**********
C     THIS SUBROUTINE INCORPORATES CHANGES NEEDED IN EVALUATING
C     THE MODAL FUNCTION ALONG THE IMAGINARY AXIS WHEN SURFACE
C     ROUGHNESS IS INCORPORATED INTO THE MODEL
C**********
      IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
      COMPLEX F,Z,PARTS
      DIMENSION PART(2)
       COMMON/TMCCOM/TMESH
       EQUIVALENCE (PART,PARTS)
C
C     DEFINE SMALL OFFSET VALUE
C
       DATA  OFFSET/1.0E-9/
C*****
C
       PART(1)=DFLOAT(JR)*TMESH
       PART(2)=DFLOAT(JI)*TMESH
C*****
C     IF EVALUATING MODAL FUNCTION ALONG IMAGINARY AXIS,
C     SHIFT ARGUMENT OF FUNCTION OFF OF AND SLIGHTLY TO THE
C     RIGHT OF THE IMAGINARY AXIS IF THE SEARCH RECTANGLE
C     LIES TO THE RIGHT OF THE IMAGINARY AXIS.
C     IF EVALUATING MODAL FUNCTION ALONG IMAGINARY AXIS,
C     SHIFT ARGUMENT OF FUNCTION OFF OF AND SLIGHTLY TO THE
C     LEFT OF THE IMAGINARY AXIS IF THE SEARCH RECTANGLE
C     LIES TO THE LEFT OF THE IMAGINARY AXIS.
C
       IF(JR.EQ.0) THEN
       IF(TLEFT.EQ.0.0) SHIFT=OFFSET
       IF(TRIGHT.EQ.0.0) SHIFT=-OFFSET
       PART(1)=SHIFT
       END IF
       Z=PARTS
       CALL FCTVLX(Z,F,FE)
       RETURN
       END
```

```
1         SUBROUTINE EXCFAC(XFAC,XFACE,Q11)
2  C**********
3  C      THIS SUBROUTINE EVALUATES THE EXCITATION FACTOR, XFAC*EXP(XFACE),
4  C      FOR THE MODE ASSOCIATED WITH THE EIGENVALUE Q11
5  C
6  C      INPUTS VIA COMMON BLOCK...
7  C         A1H - ALPHA1*H
8  C         WAVENO - WAVE NUMBER
9  C         C11 - CSRT(ALPHA1/WAVENO)
10 C
11 C      SUBROUTINES USED...
12 C          FDFDTX
13 C**********
14        IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
15        COMPLEX  XFAC,Q11,RHO,F,DFDT,CSQRT,I,Q11T
16        COMPLEX     SQNG
17        COMMON/COMTWO/ETAARS,ALPHA1,H,C11,SQNG
18        COMMON/COMTHR/Z2,Z3,A2MAG,ALPHA3,WAVENO,ALPHA2
19        DATA  I/(0.0,1.0)/
20 C*****
21 C      EVALUATE RHO
22 C
23        RHO=WAVENO*CSQRT(1.0+I*ETAARS-ALPHA1*H-C11*C11*Q11)
24 C
25 C      THE NEXT STATEMENT COMPENSATES FOR THE SHIFT IN Q11 INTRODUCED
26 C      IN SUBROUTINE 'FDFDTX'
27 C
28        Q11T=Q11+I*0.003
29 C
30 C      EVALUATE THE DERIVATIVE OF THE MODAL FUNCTION
31 C
32        CALL FDFDTX(Q11T,F,FE,DFDT,DFDTE)
33 C
34 C      EVALUATE THE EXCITATION FACTOR
35 C
36        XFAC=-WAVENO*WAVENO*C11*C11/(2.0*CSQRT(RHO)*DFDT)
37        XFACE=-DFDTE
38 C
39 C      NORMALIZE XFAC
40 C
41        XNORM=CABS(XFAC)
42        XFAC=XFAC/XNORM
43        XFACE=XFACE+DLOG(XNORM)
44        RETURN
45 C*****
46        END
```

```
1       SUBROUTINE HTGAIN(ENZA,ENZE,Q11,ZRCVR,ZXMTR)
2 C**********
3 C     THIS PROGRAM EVALUATES THE HEIGHT-GAIN FUNCTIONS FOR A
4 C     TRILINEAR MODEL OF THE MODIFIED REFRACTIVITY OF THE TROPOSPHERE.
5 C
6 C     INPUTS...
7 C        Q11 - MODAL EIGENVALUE
8 C        ZRCVR - HEIGHT (IN METERS) OF THE RECEIVER ABOVE GROUND
9 C           LEVEL
10 C          ZXMTR - HEIGHT (IN METERS) OF THE TRANSMITTER ABOVE GROUND
11 C             LEVEL
12 C
13 C     OUTPUTS...
14 C        ENZ - HEIGHT-GAIN FUNCTION FOR THE MODAL EIGENVALUE
15 C           Q11 EVALUATED FOR THE TRANSMITTER LOCATED AT ZXMTR
16 C           AND THE RECEIVER LOCATED AT ZRCVR
17 C           ENZ=ENZA*EXP(ENZE)
18 C
19 C     SUBROUTINES USED...
20 C        HNKLX,KHNKLX,CADD,KHATX
21 C**********
22      IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
23      COMPLEX  Q11,Q12,Q22,Q23,Q33,
24     $     Q12A1,Q12A2,DQ12A1,DQ12A2,Q22A1,Q22A2,DQ22A1,DQ22A2,
25     $     Q23A1,Q23A2,DQ23A1,DQ23A2,Q33A1,Q33A2,DQ33A1,DQ33A2,
26     $     SQNG,I,W,KHATA1,KHATA2,QR,QT,
27     $     QRA1,QRA2,DQRA1,DQRA2,QTA1,QTA2,DQTA1,DQTA2,
28     $     R1,R2,R3,XAA,XBA,YAA,YBA,ZAA,YCA,YDA,ZBA,FK11A,
29     $     YEA,FK12A,YFA,FK22A,ENZA,T33A1,T33A2,DT33A1,DT33A2,EPI,
30     $     TRA1,TRA2,DTRA1,DTRA2,TTA1,TTA2,DTTA1,DTTA2,DKHTA1,
31     $     DKHTA2
32      DOUBLE PRECISION*6  KHATE1,KHATE2
33      COMMON/ COMONE /C1,C3,C4,C5,C6,C7,C8,C9,SQNA2
34      COMMON / COMTWO / ETAAPS,ALPHA1,H,C11,SQNG
35      COMMON /COMTHR / Z2,Z3,A2MAG,ALPHA3,WAVENO,ALPHA2
36      DATA I/(0.0,1.0)/
37      DATA W/(0.0,-1.45749544)/
38      DATA EPI/(-0.5,0.866025404)/
39 C*****
40 C     FUNCTION DEFINITION
41 C
42      CBRT(X)=X**(1.0/3.0)
43 C*****
44 C     EVALUATE Q12,Q22,Q23,Q33
45 C
46      Q12 = Q11+C4
47      Q22 = C5*Q11+C6
48      Q23 = C5*Q11+C7
49      Q33 = C8*Q11+C9
50 C*****
51 C     EVALUATE MODIFIED HANKEL FUNCTIONS NEEDED
52 C
53      CALL KHNKLX(Q12,Q12A1,Q12E1,Q12A2,Q12E2,DQ12A1,DQ12E1,
54     $   DQ12A2,DQ12E2)
55      CALL KHNKLX(Q22,Q22A1,Q22E1,Q22A2,Q22E2,DQ22A1,DQ22E1,
```

```
56       $    DQ22A2,DQ22E2)
57        CALL KHNKLX(Q23,Q23A1,Q23E1,Q23A2,Q23E2,DQ23A1,DQ23E1,
58       $    DQ23A2,DQ23E2)
59        CALL HNKLX(Q33,T33A1,T33E1,T33A2,T33E2,DT33A1,DT33E1,
60       $    DT33A2,DT33E2)
61        CALL CADD(Q33A1,Q33E1,T33A1,T33E1,-EPI*T33A2,T33E2)
62        CALL CADD(DQ33A1,DQ33E1,DT33A1,DT33E1,-EPI*DT33A2,DT33E2)
63        Q33A2=T33A2
64        Q33E2=T33E2
65        DQ33A2=DT33A2
66        DQ33E2=DT33E2
67        CALL KHATX(Q11,KHATA1,KHATE1,KHATA2,KHATE2,DKHTA1,DKHTE1,
68       $    DKHTA2,DKHTE2)
69  C*****
70  C      DETERMINE WHICH LAYERS TRANSMITTER AND RECEIVER ARE IN
71  C      AND EVALUATE QR AND QT AND MODIFIED HANKEL FUNCTIONS
72  C
73        IF(ZRCVR.LE.Z2)THEN
74  C
75  C      RECEIVER IN FIRST LAYER
76  C
77          LRCVR=1
78          QR=Q11+ALPHA1*ZRCVR/(C11*C11)
79        CALL KHNKLX(QR,QRA1,QRE1,QRA2,QRE2,DQRA1,DQRE1,
80       $    DQRA2,DQRE2)
81        ELSE IF((ZRCVR.LE.Z3).AND.(ZRCVR.GT.Z2)) THEN
82  C
83  C      RECEIVER IN SECOND LAYER
84  C
85          LRCVR=2
86          QR=C5*Q11+C6+CRRT(WAVENO*WAVENO/(A2MAG*A2MAG))*
87       $     ALPHA2*(ZRCVR-Z2)
88        CALL KHNKLX(QR,QRA1,QRE1,QRA2,QRE2,DQRA1,DQRE1,
89       $    DQRA2,DQRE2)
90        ELSE
91  C
92  C      RECEIVER IN THIRD LAYER
93  C
94          LRCVR=3
95          QR=C8*Q11+C9+CRRT(WAVENO*WAVENO/(ALPHA3*ALPHA3))*
96       $     ALPHA3*(ZRCVR-Z3)
97        CALL HNKLX(QR,TRA1,TRE1,TRA2,TRE2,DTRA1,DTRE1,
98       $    DTRA2,DTRE2)
99        CALL CADD(QRA1,QRE1,TRA1,TRE1,-EPI*TRA2,TRE2)
100        CALL CADD(DQRA1,DQRE1,DTRA1,DTRE1,-EPI*DTRA2,DTRE2)
101        QRA2=TRA2
102        QRE2=TRE2
103        DQRA2=DTRA2
104        DQRE2=DTRE2
105        END IF
106        IF(ZXMTR.LE.Z2) THEN
107 C
108 C      TRANSMITTER IN THE FIRST LAYER
109 C
110          LXMTR=1
```

```
111           QT=Q11+ALPHA1*ZXMTR/(C11*C11)
112         CALL KHNKLX(QT,QTA1,QTE1,QTA2,QTE2,DQTA1,DQTE1,
113        $    DQTA2,DQTE2)
114         ELSE IF((ZXMTR.LE.Z3).AND.(ZXMTR.GT.Z2)) THEN
115 C
116 C       TRANSMITTER IN SECOND LAYER
117 C
118           LXMTR=2
119           QT=C5*Q11+C6+CBRT(WAVENO*WAVENO/(A2MAG*A2MAG))*
120        $     ALPHA2*(ZXMTR-Z2)
121         CALL KHNKLX(QT,QTA1,QTE1,QTA2,QTE2,DQTA1,DQTE1,
122        $    DQTA2,DQTE2)
123         ELSE
124 C
125 C       TRANSMITTER IN THIRD LAYER
126 C
127           LXMTR=3
128           QT=C8*Q11+C9+CBRT(WAVENO*WAVENO/(ALPHA3*ALPHA3))*
129        $     ALPHA3*(ZXMTR-Z3)
130         CALL HNKLX(QT,TTA1,TTE1,TTA2,TTE2,DTTA1,DTTE1,
131        $    DTTA2,DTTE2)
132         CALL CADD(QTA1,QTE1,TTA1,TTE1,-EPI*TTA2,TTE2)
133         CALL CADD(DQTA1,DQTE1,DTTA1,DTTE1,-EPI*DTTA2,DTTE2)
134         QTA2=TTA2
135         QTE2=TTE2
136         DQTA2=DTTA2
137         DQTE2=DTTE2
138         END IF
139 C*****
140 C       EVALUATE HEIGHT-GAIN FUNCTIONS
141 C**
142         IF((LRCVR.EQ.1).AND.(LXMTR.EQ.1)) THEN
143 C
144 C       RECEIVER IN FIRST LAYER --- TRANSMITTER IN FIRST LAYER
145 C
146         R1=C11*C11/(W*ALPHA1)
147         XAA=SGNA2*DQ33A2*Q23A2/C3
148         XAE=DQ33E2+Q23E2
149         XBA=Q33A2*DQ23A2
150         XBE=Q33E2+DQ23E2
151         CALL CADD(YAA,YAE,XAA,XAE,XBA,XBE)
152         XAA=SGNA2*DQ22A1*Q12A2*C1
153         XAE=DQ22E1+Q12E2
154         XBA=Q22A1*DQ12A2
155         XBE=Q22E1+DQ12E2
156         CALL CADD(YBA,YBE,XAA,XAE,XBA,XBE)
157         ZAA=YAA*YBA*KHATA2
158         ZAE=YAE+YBE*KHATE2
159         XAA=SGNA2*DQ33A2*Q23A1/C3
160         XAE=DQ33E2+Q23E1
161         XBA=Q33A2*DQ23A1
162         XBE=Q33E2+DQ23E1
163         CALL CADD(YCA,YCE,XAA,XAE,XBA,XBE)
164         XAA=-SGNA2*DQ22A2*Q12A2*C1
165         XAE= DQ22E2+Q12E2
```

```
166         XBA=-G22A2*DQ12A2
167         XBE=G22E2+DQ12E2
168         CALL CADD(YDA,YDE,XAA,XAE,XBA,XBE)
169         ZBA=YCA*YDA*KHATA2
170         ZBE=YCE+YDE+KHATE2
171         CALL CADD(FK11A,FK11E,ZAA,ZAE,ZBA,ZBE)
172 C       ZAA=-YAA*YBA*KHATA1
173 C       ZAE=YAE+YBE+KHATE1
174 C       ZBA=-YCA*YDA*KHATA1
175 C       ZBE=YCE+YDE+KHATE1
176 C       CALL CADD(FK12A,FK12E,ZAA,ZAE,ZBA,ZBE)
177         XAA=SGNA2*DQ22A1*Q12A1*C1
178         XAE=DQ22E1+Q12E1
179         XBA=Q22A1*DQ12A1
180         XBE=Q22E1+DQ12E1
181         CALL CADD(YEA,YEE,XAA,XAE,XBA,XBE)
182         ZAA=YAA*YEA*KHATA1
183         ZAE=YAE+YEE+KHATE1
184         XAA=-SGNA2*DQ22A2*Q12A1*C1
185         XAE=DQ22E2+Q12E1
186         XBA=-Q22A2*DQ12A1
187         XBE=Q22E2+DQ12E1
188         CALL CADD(YFA,YFE,XAA,XAE,XBA,XBE)
189         ZBA=YCA*YFA*KHATA1
190         ZBE=YCE+YFE+KHATE1
191         CALL CADD(FK22A,FK22E,ZAA,ZAE,ZBA,ZBE)
192 C
193         ZAA=-YAA*YEA*KHATA2
194         ZAE=YAE+YEE+KHATE2
195         ZBA=-YCA*YFA*KHATA2
196         ZBE=YCE+YFE+KHATE2
197         CALL CADD(FK12A,FK12E,ZAA,ZAE,ZBA,ZBE)
198 C
199         XAA=FK11A*QTA1*QRA1
200         XAE=FK11E+QTE1+QRE1
201         XBA=FK22A*QTA2*QRA2
202         XBE=FK22E+QTE2+QRE2
203         CALL CADD(ZAA,ZAE,XAA,XAE,XBA,XBE)
204         XAA=FK12A*QTA2*QRA1
205         XAE=FK12E+QTE2+QRE1
206         XBA=FK12A*QTA1*QRA2
207         XBE=FK12E+QTE1+QRE2
208         CALL CADD(ZBA,ZBE,XAA,XAE,XBA,XBE)
209         CALL CADD(ENZA,ENZE,ZAA,ZAE,ZBA,ZBE)
210         ENZA=R1*ENZA
211         RETURN
212         END IF
213 C**
214         IF(((LRCVR.EQ.1).AND.(LXMTR.EQ.2)).OR.((LRCVR.EQ.2).
215        $    AND.(LXMTR.EQ.1))) THEN
216 C
217 C       RECEIVER IN FIRST LAYER --- TRANSMITTER IN SECOND LAYER
218 C                                  OR
219 C       RECEIVER IN SECOND LAYER --- TRANSMITTER IN FIRST LAYER
220 C
```

```
      R1W=C11*C11/ALPHA1
      XAA=SGNA2*DQ33A2*Q23A2/C3
      XAE=DQ33E2+Q23E2
      XBA=Q33A2*DQ23A2
      XBE=Q33E2+DQ23E2
      CALL CADD(YAA,YAE,XAA,XAE,XBA,XBE)
      XAA=SGNA2*DQ33A2*Q23A1/C3
      XAE= DQ33E2+Q23E1
      XBA=Q33A2*DQ23A1
      XBE=Q33E2+DQ23E1
      CALL CADD(YBA,YBE,XAA,XAE,XBA,XBE)
      XAA=YAA*KHATA2*QTA1*QRA1
      XAE=YAE+KHATE2+QTE1+QRE1
      XBA=YBA*KHATA1*QTA2*QRA2
      XBE=YBE+KHATE1+QTE2+QRE2
      CALL CADD(ZAA,ZAE,XAA,XAE,XBA,XBE)
      IF(LRCVR.EQ.1) THEN
         XAA=-YBA*KHATA2*QTA2*QRA1
         XAE=YBE+KHATE2+QTE2+QRE1
         XBA=-YAA*KHATA1*QTA1*QRA2
         XBE=YAE+KHATE1+QTE1+QRE2
      ELSE
         XAA=-YBA*KHATA2*QTA1*QRA2
         XAE=YBE+KHATE2+QTE1+QRE2
         XBA=-YAA*KHATA1*QTA2*QRA1
         XBE=YAE+KHATE1+QTE2+QRE1
      END IF
      CALL CADD(ZBA,ZBE,XAA,XAE,XBA,XBE)
      CALL CADD(ENZA,ENZE,ZAA,ZAE,ZBA,ZBE)
      ENZA=R1W*ENZA
      RETURN
      END IF
C**
      IF(((LRCVR.EQ.1).AND.(LXMTR.EQ.3)).OR.((LRCVR.EQ.3).
     *    AND.(LXMTR.EQ.1))) THEN
C
C     RECEIVER IN FIRST LAYER --- TRANSMITTER IN THIRD LAYER
C                                    OR
C     RECEIVER IN THIRD LAYER---TRANSMITTER IN FIRST LAYER
C
      R1W=C11*C11/ALPHA1
      XAA=KHATA1*QRA2*QTA2
      XAE=KHATE1+QRE2+QTE2
      IF(LRCVR.EQ.1) THEN
         XBA=-KHATA2*QRA1*QTA2
         XBE=KHATE2+QRE1+QTE2
      ELSE
         XBA=-KHATA2*QTA1*QRA2
         XBE=KHATE2+QTE1+QRE2
      END IF
      CALL CADD(ENZA,ENZE,XAA,XAE,YBA,XBE)
      ENZA=-R1W*W*ENZA
      RETURN
      END IF
C**
```

```
      IF((LRCVR.EQ.2).AND.(LXMTR.EQ.2)) THEN
C
C     RECEIVER IN SECOND LAYER --- TRANSMITTER IN SECOND LAYER
C
      R2=CBRT(ALPHA2*ALPHA2/(WAVENO*WAVENO))/(ALPHA2*H)
      IF(ALPHA2.LT.0.0)THEN
      XAA=DG33A2*Q23A2/C3
      XAE=DG33E2+Q23E2
      XBA=Q33A2*DQ23A2
      XBE=Q33E2+DQ23E2
      CALL CADD(YAA,YAE,XAA,XAE,XBA,XBE)
      XAA=C1*DQ22A2*Q12A2
      XAE=DG22E2+Q12E2
      XBA=Q22A2*DQ12A2
      XBE=Q22E2+DQ12E2
      CALL CADD(YBA,YBE,XAA,XAE,XBA,XBE)
      XAA=C1*DQ22A2*Q12A1
      XAE=DG22E2+Q12E1
      XBA=Q22A2*DQ12A1
      XBE=Q22E2+DQ12E1
      CALL CADD(YCA,YCE,XAA,XAE,XBA,XBE)
      XAA=DG33A2*Q23A1/C3
      XAE=DG33E2+Q23E1
      XBA=Q33A2*DQ23A1
      XBE=Q33E2+DQ23E1
      CALL CADD(YDA,YDE,XAA,XAE,XBA,XBE)
      XAA=C1*DQ22A1*Q12A2
      XAE=DG22E1+Q12E2
      XBA=Q22A1*DQ12A2
      XBE=Q22E1+DQ12E2
      CALL CADD(YEA,YEE,XAA,XAE,XBA,XBE)
      XAA=C1*DQ22A1*Q12A1
      XAE=DG22E1+Q12E1
      XBA=Q22A1*DQ12A1
      XBE=Q22E1+DQ12E1
      CALL CADD(YFA,YFE,XAA,XAE,XBA,XBE)
      XAA=YBA*KHATA1
      XAE=YBE+KHATE1
      XBA=-YCA*KHATA2
      XBE=YCE+KHATE2
      CALL CADD(ZAA,ZAE,XAA,XAE,XBA,XBE)
      FK11A=ZAA*YAA
      FK11E=ZAE+YAE
      FK12A=-ZAA*YDA
      FK12E=ZAE+YDE
      XAA=YFA*KHATA1
      XAE=YFE+KHATE1
      XBA=-YFA*KHATA2
      XBE=YFE+KHATE2
      CALL CADD(ZAA,ZAE,XAA,XAE,XBA,XBE)
      FK22A=YDA*ZAA
      FK22E=YDE+ZAE
      XAA=FK11A*GTA1*QRA1
      XAE=FK11E+GTE1+QRE1
      XBA=FK22A*GTA2*QRA2
```

```
      XBE=FK22E+QTE2+QRE2
      CALL CADD(ZAA,ZAE,XAA,XAE,XBA,XBE)
      XAA=FK12A*QTA2*QRA1
      XAE=FK12E+QTE2+QRE1
      XBA =FK12A*QTA1*QRA2
      XBE=FK12E+QTE1+QRE2
      CALL CADD(ZBA,ZBE,XAA,XAE,XBA,XBE)
      CALL CADD(ENZA,ENZE,ZAA,ZAE,ZBA,ZBE)
      ENZA=R2*ENZA
      RETURN
      ELSE
      XAA=-DQ33A2*Q23A2/C3
      XAE=DQ33E2+Q23E2
      XBA=Q33A2*DQ23A2
      XBE=Q33E2+DQ23E2
      CALL CADD(YAA,YAE,XAA,XAE,XBA,XBE)
      XAA=-C1*DQ22A2*Q12A2
      XAE=DQ22E2+Q12E2
      XBA=Q22A2*DQ12A2
      XBE=Q22E2+DQ12E2
      CALL CADD(YBA,YBE,XAA,XAE,XBA,XBE)
      XAA=C1*DQ22A2*Q12A1
      XAE=DQ22E2+Q12E1
      XBA=-Q22A2*DQ12A1
      XBE=Q22E2+DQ12E1
      CALL CADD(YCA,YCE,XAA,XAE,XBA,XBE)
      XAA=C1*DQ22A1*Q12A2
      XAE=DQ22E1+Q12E2
      XBA=-Q22A1*DQ12A2
      XBE=Q22E1+DQ12E2
      CALL CADD(YDA,YDE,XAA,XAE,XBA,XBE)
      XAA=-C1*DQ22A1*Q12A1
      XAE=DQ22E1+Q12E1
      XBA=Q22A1*DQ12A1
      XBE=Q22E1+DQ12E1
      CALL CADD(YEA,YEE,XAA,XAE,XBA,XBE)
      XAA=-DQ33A2*Q23A1/C3
      XAE=DQ33E2+Q23E1
      XBA=Q33A2*DQ23A1
      XBE=Q33E2+DQ23E1
      CALL CADD(YFA,YFE,XAA,XAE,XBA,XBE)
      XAA=YBA*KHATA1
      XAE=YBE+KHATE1
      XBA=YCA*KHATA2
      XBE=YCE+KHATE2
      CALL CADD(ZAA,ZAE,XAA,XAE,XBA,XBE)
      FK11A=YAA*ZAA
      FK11E=YAE+ZAE
      XAA=YDA*KHATA1
      XAE=YDE+KHATE1
      XBA=YFA*KHATA2
      XBE=YFE+KHATE2
      CALL CADD(ZAA,ZAE,XAA,XAE,XBA,XBE)
      FK12A=YAA*ZAA
      FK12E=YAE+ZAE
```

```
      FK22A=-YFA*ZAA
      FK22E=YFE+ZAE
      XAA=FK11A*GTA1*GRA1
      XAE=FK11E+GTE1+GRE1
      XBA=FK22A*GTA2*GRA2
      XBE=FK22E+GTE2+GRE2
      CALL CADD(ZAA,ZAE,XAA,XAE,XBA,XBE)
      XAA=FK12A*GTA2*GRA1
      XAE=FK12E+GTE2+GRE1
      XBA=FK12A*GTA1*GRA2
      XBE=FK12E+GTE1+GRE2
      CALL CADD(ZBA,ZBE,XAA,XAE,XBA,XBE)
      CALL CADD(EN7A,ENZE,ZAA,ZAE,ZBA,ZBE)
      EN7A=R2*EN7A
      RETURN
      END IF
      END IF
C**
      IF(((LRCVR.EQ.2).AND.(LXMTR.EQ.3)).OR.((LRCVR.EQ.3).
     &    AND.(LXMTR.EQ.2))) THEN
C
C     RECEIVER IN SECOND LAYER---TRANSMITTER IN THIRD LAYER
C                                 OR
C     RECEIVER IN THIRD LAYER---TRANSMITTER IN SECOND LAYER
C
      R2W=CBRT(A2MAG*A2MAG/(WAVENO*WAVENO))/ALPHA2
      XAA=SGNA2*C1*DQ22A2*Q12A2
      XAE=DQ22E2+Q12E2
      XBA=Q22A2*DQ12A2
      XBE=Q22E2+DQ12E2
      CALL CADD(YAA,YAE,XAA,XAE,XBA,XBE)
      XAA=SGNA2*DQ22A2*Q12A1*C1
      XAE=DQ22E2+Q12E1
      XBA=Q22A2*DQ12A1
      XBE=Q22E2+DQ12E1
      CALL CADD(YBA,YBE,XAA,XAE,XBA,XBE)
      XAA=SGNA2*C1*DQ22A1*Q12A2
      XAE=DQ22E1+Q12E2
      XBA=Q22A1*DQ12A2
      XBE=Q22E1+DQ12E2
      CALL CADD(YCA,YCE,XAA,XAE,XBA,XBE)
      XAA=SGNA2*C1*DQ22A1*Q12A1
      XAE=DQ22E1+Q12E1
      XBA=Q22A1*DQ12A1
      XBE=Q22E1+DQ12E1
      CALL CADD(YDA,YDE,XAA,XAE,XBA,XBE)
      XAA=YAA*KHATA1
      XAE=YAE+KHATE1
      XBA=-YBA*KHATA2
      XBE=YBE+KHATE2
      CALL CADD(FK12A,FK12E,XAA,XAE,XBA,XBE)
      XAA=-YCA*KHATA1
      XAE=YCE+KHATE1
      XBA=YDA*KHATA2
      XBE=YDE+KHATE2
```

```
      CALL CADD(FK22A,FK22E,XAA,XAE,XBA,XBE)
      IF(LRCVR.EQ.2) THEN
         ZAA=FK12A*QTA2*QRA1
         ZAE=FK12E+QTE2+QRE1
         ZBA=FK22A*QTA2*QRA2
         ZBE=FK22E+QTE2+QRE2
         CALL CADD(ENZA,ENZE,ZAA,ZAE,ZBA,ZBE)
      ELSE
         ZAA=FK12A*QTA1*QRA2
         ZAE=FK12E+QTE1+QRE2
         ZBA=FK22A*QTA2*QRA2
         ZBE=FK22E+QTE2+QRE2
         CALL CADD(ENZA,ENZE,ZAA,ZAE,ZBA,ZBE)
      END IF
      ENZA=R2W*ENZA
      RETURN
      END IF
C**
      IF((LRCVR.EQ.3).AND.(LXMTR.EQ.3)) THEN
C
C     RECEIVER IN THIRD LAYER--- TRANSMITTER IN THIRD LAYER
C
      R3=CSRT(ALPHA3*ALPHA3/(WAVENO*WAVENO))/(ALPHA3*W)
      XAA=SGNA2*C1*DQ22A1*Q12A2
      XAE=DQ22E1+Q12E2
      XBA=Q22A1*DQ12A2
      XBE=Q22E1+DQ12E2
      CALL CADD(YAA,YAE,XAA,XAE,XBA,XBE)
      XAA=SGNA2*Q23A2*DQ33A1/C3
      XAE=Q23E2+DQ33E1
      XBA=DQ23A2*Q33A1
      XBE=DQ23E2+Q33E1
      CALL CADD(YBA,YBE,XAA,XAE,XBA,XBE)
      XAA=SGNA2*C1*DQ22A2*Q12A2
      XAE=DQ22E2+Q12E2
      XBA=Q22A2*DQ12A2
      XBE=Q22E2+DQ12E2
      CALL CADD(YCA,YCE,XAA,XAE,XBA,XBE)
      XAA=SGNA2*DQ33A1*Q23A1/C3
      XAE=DQ33E1+Q23E1
      XBA=Q33A1*DQ23A1
      XBE=Q33E1+DQ23E1
      CALL CADD(YDA,YDE,XAA,XAE,XBA,XBE)
      XAA=SGNA2*C1*DQ22A1*Q12A1
      XAE=DQ22E1+Q12E1
      XBA=Q22A1*DQ12A1
      XBE=Q22E1+DQ12E1
      CALL CADD(YEA,YEE,XAA,XAE,XBA,XBE)
      XAA=SGNA2*C1*DQ22A2*Q12A1
      XAE=DQ22E2+Q12E1
      XBA=Q22A2*DQ12A1
      XBE=Q22E2+DQ12E1
      CALL CADD(YFA,YFE,XAA,XAE,XBA,XBE)
      XAA=-YAA*YBA*KHATA1
      XAE=YAE+YBE*KHATE1
```

```
      XBA=YCA*YDA*KHATA1
      XBE=YCE+YDE+KHATE1
      CALL CADD(ZAA,ZAE,XAA,XAE,XBA,XBE)
      XAA=YEA*YBA*KHATA2
      XAE=YEE+YBE+KHATE2
      XBA=-YFA*YDA*KHATA2
      XBE=YFE+YDE+KHATE2
      CALL CADD(ZBA,ZBE,XAA,XAE,XBA,XBE)
      CALL CADD(FK22A,FK22E,ZAA,ZAE,ZBA,ZBE)
      ENZA=R3*QTA2*QRA2*FK22A
      ENZE=QTE2+QRE2+FK22E
      RETURN
      END IF
C*****
      END
```

```
1       SUBROUTINE MODSUM(ECMS,EIMS,RNG,ZRCVR,ZXMTR,ZEROS,NRZ,NRMODE)
2 C**********
3 C     THIS PROGRAM EVALUATES THE FIELD STRENGTH IN A TRILINEAR
4 C     MODEL OF THE MODIFIED REFRACTIVITY OF THE TROPOSPHERE RELATIVE TO
5 C     THE FREE SPACE VALUE. THE FIELD IS HORIZONTALLY POLARIZED.
6 C
7 C     INPUTS...
8 C        RNG - RANGE(IN KILOMETERS) AT WHICH THE FIELD STRENGTH
9 C           IS TO BE EVALUATED.
10 C          ZRCVR - HEIGHT (IN METERS) AT WHICH THE FIELD STRENGTH
11 C             IS TO BE EVALUATED.
12 C          ZXMTR - HEIGHT (IN METERS) OF THE TRANSMITTER
13 C          ZEROS - ARRAY CONTAINING THE MODAL EIGENVALUES Q11
14 C          NRZ - DIMENSION OF ARRAY 'ZEROS'
15 C          NRMODE - NUMBER OF MODAL EIGENVALUES CONTAINED IN ARRAY 'ZEROS'
16 C
17 C     OUTPUTS...
18 C        ECMS - COHERENT MODE SUM FIELD STRNGTH (IN DB) RELATIVE
19 C           TO FREE SPACE VALUE
20 C        EIMS - INCOHERENT MODE SUM FIELD STRENGTH (IN DB) RELATIVE
21 C           TO FREE SPACE VALUE
22 C
23 C     SUBROUTINES USED....
24 C        EXCFAC,HTGAIN,CADD,ADDX
25 C**********
26        IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
27        COMPLEX  ZEROS,XFAC,ENZA,CEXP,I,RHO,CSQRT,HFACA,SUMCA,XAA,XBA
28        COMPLEX  SQNG
29        DIMENSION  ZEROS(NRZ)
30        COMMON/COMTWO/ETAABS,ALPHA1,H,C11,SQNG
31        COMMON/COMTHR/Z2,Z3,A2MAG,ALPHA3,WAVENO,ALPHA2
32        DATA I/(0.0,1.0)/
33        DATA PI/3.14159265/
34        DATA ELOG10/0.434294482/
35 C*****
36 C     INITIALIZE
37 C
38        RNGM=RNG*1000.0
39        BETA=2.0*SQRT(2.0*PI*RNGM)
40        A1H=1.0-ALPHA1*H
41        C11SQ=C11*C11
42        SUMCA=(0.0,0.0)
43        SUMCE=0.0
44        XBA=(0.0,0.0)
45        XBE=0.0
46        SUMIA=0.0
47        SUMIE=0.0
48        XDA=0.0
49        XDE=0.0
50 C
51 C     EVALUATE MODE SUMS
52 C
53        DO 100 JMODE=1,NRMODE
54        RHO=WAVENO*CSQRT(A1H+I*ETAABS-C11SQ*ZEROS(JMODE))
55        HFACA=CEXP(-I*(REAL(RHO)-WAVENO)*RNGM)
```

```
      HFACE=AIMAG(RHO)*RNGM
      CALL EXCFAC(XFAC,XFACE,ZEROS(JMODE))
      CALL HTGAIN(ENZA,ENZE,ZEROS(JMODE),ZRCVR,ZXMTR)
      XAA=XFAC*ENZA*HFACA
      XAE=XFACE*ENZE*HFACE
C
C     COHERENT MODE SUM
C
      CALL CADD(SUMCA,SUMCE,XAA,XAE,XBA,XBE)
      XBA=SUMCA
      XBE=SUMCE
C
C     INCOHERENT MODE SUM
C
      XCA=CABS(XAA)*CABS(XAA)
      XCE=2.0*XAE
      CALL ADDX(SUMIA,SUMIE,XCA,XCE,XDA,XDE)
      XDA=SUMIA
      XDE=SUMIE
  100 CONTINUE
C
C     CALCULATE COHERENT MODE SUM FIELD STRENGTH
C
      ECMS=20.0*(DLOG10(BETA*CABS(SUMCA))+SUMCE*ELOG10)
C
C     CALCULATE INCOHERENT MODE SUM FIELD STRENGTH
C
      EIMS=20.0*(DLOG10(BETA*SQRT(SUMIA))+0.5*SUMIE*ELOG10)
      RETURN
C*****
      END
```

```
1       SUBROUTINE SEAH2O(SIGEFF,EPSEFF,T,S,FREQ)
2 C**********
3 C     THIS PROGRAM EVALUATES THE REAL EFFECTIVE RELATIVE
4 C     DIELECTRIC CONSTANT AND THE REAL EFFECTIVE CONDUCTIVITY
5 C     OF SEAWATER AS A FUNCTION OF TEMPERATURE, SALINITY, AND
6 C     FREQUENCY.  THIS PROGRAM IS BASED ON THE PAPER:
7 C     L.A. KLEIN AND C.T. SWIFT "AN IMPROVED MODEL FOR THE
8 C     DIELECTRIC CONSTANT OF SEA WATER AT MICROWAVE
9 C     FREQUENCIES"  IEEE TRANS. ANT. AND PROP., VOL. 25 (1977)
10 C     104-111.
11 C
12 C     NOTE:  THE VALUES THIS PROGRAM GIVES FOR EPSEFF AND
13 C            SIGEFF AS THE SALINITY APPROACHES ZERO ARE NOT
14 C            CONTINUOUS WITH THE VALUES GIVEN WHEN THE SALINITY
15 C            EQUALS ZERO.  SEE THE PAPER BY KLEIN AND SWIFT.
16 C
17 C     INPUTS...
18 C        T - TEMPERATURE OF SEAWATER (IN DEGREES CELSIUS)
19 C        S - SALINITY OF SEAWATER (IN GRAMS SALT/KG SEAWATER)
20 C        FREQ - FREQUENCY (IN HERTZ)
21 C
22 C     OUTPUTS...
23 C        SIGEFF - THE REAL EFFECTIVE CONDUCTIVITY (IN SI/M)
24 C        EPSEFF - THE REAL EFFECTIVE RELATIVE DIELECTRIC CONSTANT
25 C
26 C     NOTE:  THE COMPLEX DIELECTRIC CONSTANT, EPS, IS EQUAL TO
27 C                  EPS=EPSEFF-I*SIGEFF/(2*PI*FREQ*EPS0)
28 C            WHERE EPS0 IS THE PERMITTIVITY OF FREE SPACE
29 C**********
30       IMPLICIT DOUBLE PRECISION*6 (A-H,O-Z)
31       DATA PI/3.1415926536/
32       DATA EPS0/8.85434E-12/
33 C
34       OMEGA=2.0*PI*FREQ
35 C*****
36 C     CALCULATE THE STATIC DIELECTRIC CONSTANT
37 C
38       IF(S.GT.0.0) THEN
39       ET=87.134+T*(-0.1949+T*(-1.276E-2+2.491E-4*T))
40       AST=1.0+1.613E-5*S*T+S*(-3.656E-3+S*(3.210E-5-4.232E-7*S))
41       ES=ET*AST
42       ELSE
43       ES=88.045+T*(-0.4147+T*(6.295E-4+1.075E-5*T))
44       END IF
45 C*****
46 C     CALCULATE THE RELAXATION TIME
47 C
48       TT=1.768E-11+T*(-6.086E-13+T*(1.104E-14-8.111E-17*T))
49       BST=1.0+2.282E-5*S*T+S*(-7.638E-4+S*(-7.760E-6+1.105E-8*S))
50       TAU=TT*BST
51 C*****
52 C     CALCULATE THE IONIC CONDUCTIVITY
53 C
54       DEL=25.0-T
55       BETA=2.033E-2+DEL*(1.266E-4+2.464E-6*DEL)-S*(1.849E-5
```

```
56      $   +DEL*(-2.551E-7+2.551E-8*DEL))
57          SIG=S*(0.182521+S*(-1.46192E-3+S*(2.09324E-5-1.28205E-7*S)))
58          SIGMA=SIG*DEXP(-DEL*BETA)
59  C*****
60  C       CALCULATE THE REAL EFFECTIVE RELATIVE DIELECTRIC CONSTANT
61  C
62          EPSINF=4.9
63          FAC=(ES-EPSINF)/(1.0+(OMEGA*TAU)**2)
64          EPSEFF=EPSINF+FAC
65  C*****
66  C       CALCULATE THE REAL EFFECTIVE CONDUCTIVITY
67  C
68          SIGEFF=OMEGA*OMEGA*TAU*EPSO*FAC+SIGMA
69  C*****
70          RETURN
71          END
```

```
      SUBROUTINE DHORIZ(DHZ,ZRCVR,ZXMTR)
C**********
C     THIS PROGRAM CALCULATES THE HORIZON DISTANCE FOR A RECEIVER
C     AT HEIGHT ZRCVR AND A TRANSMITTER AT HEIGHT ZXMTR
C
C     INPUTS...
C        ZRCVR - RECEIVER HEIGHT (IN METERS)
C        ZXMTR - TRANSMITTER HEIGHT (IN METERS)
C
C     OUTPUTS...
C        DHZ - HORIZON DISTANCE (IN KILOMETERS)
C**********
      IMPLICIT DOUBLE PRECISION*8 (A-H,O-Z)
C
C     RADIUS OF EARTH (IN METERS)
C
      DATA  R/6.378160E6/
C*****
C     CALCULATE EFFECTIVE EARTH RADIUS
C
      REFF=4.0*R/3.0
C*****
C     CALCULATE APPROXIMATE HORIZON DISTANCE FOR RECEIVER AND
C     TRANSMITTER
C
      HRCVR=DSQRT(2.0*REFF*ZRCVR)
      HXMTR=DSQRT(2.0*REFF*ZXMTR)
C*****
C     COMBINED HORIZON DISTANCE IN KILOMETERS
C
      DHZ=(HRCVR+HXMTR)*1.0E-3
      RETURN
      END
```

```
1         FUNCTION CASIN(Z)
2  C**********
3  C      THIS SUBPROGRAM COMPUTES THE COMPLEX ARCSINE OF THE COMPLEX
4  C      NUMBER Z
5  C**********
6         COMPLEX  CASIN,Z,CLOG,CSQRT,I
7         DATA I/(0.0,1.0)/
8         CASIN=-I*CLOG(CSQRT(1.0-Z*Z)+I*Z)
9         RETURN
10         END
```

10. SAMPLE OUTPUT FROM XWVG

```
FREQ =     520.0000 MHZ
ALPHA1 =      0.2360E-06 /M
ALPHA2 =     -0.6496E-06 /M
ALPHA3 =      0.5184E-06 /M
Z2 =       182.9000 M
Z3 =       304.8000 M
ALOSS =         1.0000 DB/KM

 M(  0.00) = 341.00

 HORIZONTAL POLARIZATION

 SEAWATER TEMPERATURE =      1.6000E+01 DEGREES CELSIUS
 SEAWATER SALINITY =      3.5000E+01 GRAMS SALT/KG SEAWATER
 DELTA =      0.0000E+00 M
 EPSGND =      6.5257E-10 FARADS/M
 SGMGND =      4.4544E+00 MHOS/M
 ETA =      0.0000E+00




      ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
      TLEFT =     0.00      TRIGHT =     2.00      TBOT =     0.00      TTOP =     2

                FIGEN(  1) =      1.0946E+00     1.5815E-04



      ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
      TLEFT =    -2.00      TRIGHT =     0.00      TBOT =     0.00      TTOP =     2

                FIGEN(  1) =     -1.3124E+00     8.8007E-06
                FIGEN(  2) =     -1.4630E-01     3.9252E-05



      ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
      TLEFT =    -4.00      TRIGHT =    -2.00      TBOT =     0.00      TTOP =     2

                FIGEN(  1) =     -2.5895E+00     4.6858E-07
```

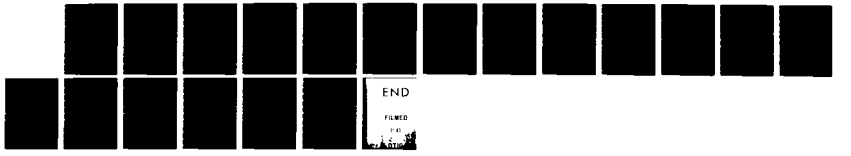
```
ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =   -6.00     TRIGHT =   -4.00     TBOT =    0.00     TTOP =    2

          EIGEN(  1) =    -4.2052E+00    2.2410E-09


ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =   -8.00     TRIGHT =   -6.00     TBOT =    0.00     TTOP =    2


ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =  -10.00     TRIGHT =   -8.00     TBOT =    0.00     TTOP =    2


ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =  -12.00     TRIGHT =  -10.00     TBOT =    0.00     TTOP =    2


ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =  -14.00     TRIGHT =  -12.00     TBOT =    0.00     TTOP =    2


ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =  -16.00     TRIGHT =  -14.00     TBOT =    0.00     TTOP =    2


ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =    2.00     TRIGHT =    4.00     TBOT =    0.00     TTOP =    2

          EIGEN(  1) =     2.4514E+00    3.6817E-03
          EIGEN(  2) =     3.8044E+00    5.8611E-02
```

```
?EROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =    4.00      TRIGHT =    6.00      TBOT =    0.00      TTOP =    2

          EIGEN(  1) =     5.2977E+00     2.9019E-01


?EROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =    6.00      TRIGHT =    8.00      TBOT =    0.00      TTOP =    2

          EIGEN(  1) =     7.0447E+00     6.3569E-01


?EROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =    8.00      TRIGHT =   10.00      TBOT =    0.00      TTOP =    2

          EIGEN(  1) =     9.1272E+00     9.7805E-01


?EROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =   10.00      TRIGHT =   12.00      TBOT =    0.00      TTOP =    2

          EIGEN(  1) =     1.1421E+01     1.3358E+00


?EROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =   12.00      TRIGHT =   14.00      TBOT =    0.00      TTOP =    2

          EIGEN(  1) =     1.3989E+01     1.6259E+00


?EROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =   14.00      TRIGHT =   16.00      TBOT =    0.00      TTOP =    2

          EIGEN(  1) =     1.3989E+01     1.6259E+00
```

```
ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =   16.00     TRIGHT =   18.00     TBOT =    0.00     TTOP =    2

          EIGEN(  1) =     1.6795E+01    1.9917E+00


ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =   18.00     TRIGHT =   20.00     TBOT =    0.00     TTOP =    2

          EIGEN(  1) =     1.9789E+01    2.2808E+00


ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =   20.00     TRIGHT =   22.00     TBOT =    0.00     TTOP =    2


ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =   22.00     TRIGHT =   24.00     TBOT =    0.00     TTOP =    2

          EIGEN(  1) =     2.3103E+01    2.6178E+00


ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =   24.00     TRIGHT =   26.00     TBOT =    0.00     TTOP =    2


ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT =   26.00     TRIGHT =   28.00     TBOT =    0.00     TTOP =    2


ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
```

AD-A133 667 XWVG: A WAVEGUIDE PROGRAM FOR TRILINEAR TROPOSPHERIC DUCTS COMPUTER PROGR.. (U) NAVAL OCEAN SYSTEMS CENTER SAN DIEGO CA G B BAUMGARTNER 30 JUN 83 NOSC/TD-610 3/3

UNCLASSIFIED F/G 20/14 NL



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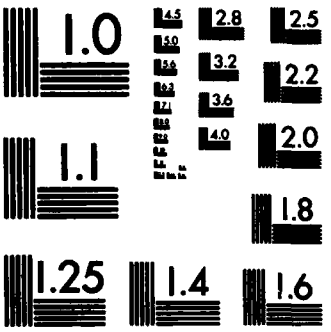

TLEFT = 28.00 TRIGHT = 30.00 TBCT = 0.00 TTCP = 2

ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT = 30.00 TRIGHT = 32.00 TBCT = 0.00 TTCP = 2

ZEROS FOUND IN EXPANDED SEARCH RECTANGLE DEFINED BY
TLEFT = 32.00 TRIGHT = 34.00 TBCT = 0.00 TTCP = 2

SOLUTIONS OF THE MODAL EQUATION

Q11(1) =	-4.2052E+00	2.2410E-09	THETA(1) =	3.2552E-13	2.67
Q11(2) =	-2.5895E+00	4.6858E-07	THETA(2) =	6.8669E-11	2.64
Q11(3) =	-1.3124E+00	8.8007E-06	THETA(3) =	1.2989E-09	2.63
Q11(4) =	-1.4630E-01	3.9252E-05	THETA(4) =	5.8317E-09	2.61
Q11(5) =	1.0946E+00	1.5815E-04	THETA(5) =	2.3665E-08	2.59
Q11(6) =	2.4514E+00	3.6817E-03	THETA(6) =	5.5526E-07	2.57
Q11(7) =	3.8044E+00	5.8611E-02	THETA(7) =	8.9105E-06	2.55
Q11(8) =	5.2977E+00	2.9019E-01	THETA(8) =	4.4515E-05	2.53
Q11(9) =	7.0447E+00	6.3569E-01	THETA(9) =	9.8564E-05	2.50
Q11(10) =	9.1272E+00	9.7805E-01	THETA(10) =	1.5364E-04	2.47
Q11(11) =	1.1421E+01	1.3358E+00	THETA(11) =	2.1297E-04	2.43
Q11(12) =	1.3989E+01	1.6259E+00	THETA(12) =	2.6368E-04	2.39
Q11(13) =	1.6795E+01	1.9917E+00	THETA(13) =	3.2932E-04	2.34
Q11(14) =	1.9789E+01	2.2808E+00	THETA(14) =	3.8532E-04	2.29
Q11(15) =	2.3103E+01	2.6178E+00	THETA(15) =	4.5343E-04	2.24

FIELD STRENGTH RELATIVE TO FREE SPACE VALUE

RANGE (KM)	ZXMTR (M)	ZRCVR (M)	COHERENT MODE SUM (DB)	INCOHERENT MODE SUM (DB)	HORIZON DISTANCE (KM)
185.32	30.480	0.000	-68.7303	-57.8077	22.77
185.32	30.480	2.000	-19.6558	-8.7228	28.60
185.32	30.480	4.000	-13.7076	-2.7435	31.02

185.32	30.480	6.000	-10.2968	0.7192	32.87
185.32	30.480	8.000	-7.9531	3.1361	34.43
185.32	30.480	10.000	-6.2162	4.9679	35.81
185.32	30.480	12.000	-4.8815	6.4194	37.06
185.32	30.480	14.000	-3.8410	7.5994	38.20
185.32	30.480	16.000	-3.0307	8.5726	39.27
185.32	30.480	18.000	-2.4098	9.3802	40.27
185.32	30.480	20.000	-1.9510	10.0503	41.21
185.32	30.480	22.000	-1.6348	10.6024	42.11
185.32	30.480	24.000	-1.4466	11.0509	42.97
185.32	30.480	26.000	-1.3746	11.4064	43.80
185.32	30.480	28.000	-1.4074	11.6771	44.59
185.32	30.480	30.000	-1.5330	11.8693	45.36
185.32	30.480	32.000	-1.7360	11.9882	46.10
185.32	30.480	34.000	-1.9951	12.0383	46.82
185.32	30.480	36.000	-2.2798	12.0234	47.51
185.32	30.480	38.000	-2.5512	11.9473	48.19
185.32	30.480	40.000	-2.7582	11.8141	48.85
185.32	30.480	42.000	-2.8472	11.6283	49.50
185.32	30.480	44.000	-2.7757	11.3957	50.13
185.32	30.480	46.000	-2.5264	11.1233	50.74
185.32	30.480	48.000	-2.1143	10.8204	51.34
185.32	30.480	50.000	-1.5787	10.4984	51.93
185.32	30.480	52.000	-0.9676	10.1715	52.51
185.32	30.480	54.000	-0.3248	9.8562	53.07
185.32	30.480	56.000	0.3167	9.5705	53.63
185.32	30.480	58.000	0.9352	9.3314	54.18
185.32	30.480	60.000	1.5183	9.1532	54.71
185.32	30.480	62.000	2.0600	9.0440	55.24
185.32	30.480	64.000	2.5587	9.0042	55.76
185.32	30.480	66.000	3.0153	9.0266	56.27
185.32	30.480	68.000	3.4318	9.0978	56.78
185.32	30.480	70.000	3.8103	9.2011	57.27
185.32	30.480	72.000	4.1528	9.3193	57.76
185.32	30.480	74.000	4.4602	9.4366	58.25
185.32	30.480	76.000	4.7322	9.5399	58.72
185.32	30.480	78.000	4.9672	9.6193	59.19
185.32	30.480	80.000	5.1623	9.6680	59.66
185.32	30.480	82.000	5.3132	9.6821	60.11
185.32	30.480	84.000	5.4143	9.6604	60.57
185.32	30.480	86.000	5.4590	9.6039	61.01
185.32	30.480	88.000	5.4396	9.5161	61.46
185.32	30.480	90.000	5.3476	9.4021	61.89
185.32	30.480	92.000	5.1737	9.2693	62.33
185.32	30.480	94.000	4.9079	9.1263	62.75
185.32	30.480	96.000	4.5401	8.9828	63.18
185.32	30.480	98.000	4.0608	8.8487	63.60
185.32	30.480	100.000	3.4633	8.7331	64.01
185.32	30.480	102.000	2.7492	8.6430	64.42
185.32	30.480	104.000	1.9395	8.5827	64.83
185.32	30.480	106.000	1.0961	8.5528	65.23
185.32	30.480	108.000	0.3532	8.5503	65.63
185.32	30.480	110.000	-0.0739	8.5693	66.02
185.32	30.480	112.000	0.0218	8.6020	66.41
185.32	30.480	114.000	0.6666	8.6398	66.80

185.32	30.480	116.000	1.6773	8.6745	67.19
185.32	30.480	118.000	2.8228	8.6991	67.57
185.32	30.480	120.000	3.9438	8.7081	67.95
185.32	30.480	122.000	4.9591	8.6984	68.32
185.32	30.480	124.000	5.8347	8.6686	68.69
185.32	30.480	126.000	6.5594	8.6196	69.06
185.32	30.480	128.000	7.1312	8.5544	69.43
185.32	30.480	130.000	7.5505	8.4774	69.79
185.32	30.480	132.000	7.8178	8.3946	70.15
185.32	30.480	134.000	7.9315	8.3123	70.51
185.32	30.480	136.000	7.8879	8.2372	70.86
185.32	30.480	138.000	7.6802	8.1751	71.22
185.32	30.480	140.000	7.2993	8.1301	71.57
185.32	30.480	142.000	6.7340	8.1044	71.91
185.32	30.480	144.000	5.9749	8.0976	72.26
185.32	30.480	146.000	5.0246	8.1071	72.60
185.32	30.480	148.000	3.9254	8.1283	72.94
185.32	30.480	150.000	2.8227	8.1557	73.28
185.32	30.480	152.000	2.0505	8.1834	73.61
185.32	30.480	154.000	2.0454	8.2059	73.95
185.32	30.480	156.000	2.9106	8.2186	74.28
185.32	30.480	158.000	4.2767	8.2184	74.61
185.32	30.480	160.000	5.7398	8.2040	74.94
185.32	30.480	162.000	7.0905	8.1751	75.26
185.32	30.480	164.000	8.2566	8.1337	75.58
185.32	30.480	166.000	9.2248	8.0824	75.90
185.32	30.480	168.000	10.0009	8.0253	76.22
185.32	30.480	170.000	10.5953	7.9667	76.54
185.32	30.480	172.000	11.0168	7.9109	76.86
185.32	30.480	174.000	11.2720	7.8616	77.17
185.32	30.480	176.000	11.3637	7.8213	77.48
185.32	30.480	178.000	11.2919	7.7909	77.79
185.32	30.480	180.000	11.0534	7.7699	78.10
185.32	30.480	182.000	10.6430	7.7561	78.41
185.32	30.480	184.000	10.0548	7.7467	78.71
185.32	30.480	186.000	9.2906	7.7416	79.01
185.32	30.480	188.000	8.3716	7.7441	79.32
185.32	30.480	190.000	7.3596	7.7568	79.62
185.32	30.480	192.000	6.4019	7.7822	79.91
185.32	30.480	194.000	5.7635	7.8220	80.21
185.32	30.480	196.000	5.7329	7.8775	80.51
185.32	30.480	198.000	6.3683	7.9494	80.80
185.32	30.480	200.000	7.4356	8.0383	81.09
185.32	30.480	202.000	8.6476	8.1439	81.38
185.32	30.480	204.000	9.8260	8.2659	81.67
185.32	30.480	206.000	10.8939	8.4032	81.96
185.32	30.480	208.000	11.8288	8.5548	82.25
185.32	30.480	210.000	12.6314	8.7193	82.53
185.32	30.480	212.000	13.3109	8.8949	82.82
185.32	30.480	214.000	13.8787	9.0796	83.10
185.32	30.480	216.000	14.3463	9.2713	83.38
185.32	30.480	218.000	14.7239	9.4675	83.66
185.32	30.480	220.000	15.0208	9.6658	83.94
185.32	30.480	222.000	15.2449	9.8635	84.22
185.32	30.480	224.000	15.4035	10.0577	84.49

185.32	30.480	226.000	15.5028	10.2458	84.77
185.32	30.480	228.000	15.5482	10.4250	85.04
185.32	30.480	230.000	15.5448	10.5928	85.31
185.32	30.480	232.000	15.4969	10.7468	85.59
185.32	30.480	234.000	15.4085	10.8846	85.86
185.32	30.480	236.000	15.2831	11.0044	86.12
185.32	30.480	238.000	15.1240	11.1042	86.39
185.32	30.480	240.000	14.9340	11.1825	86.66
185.32	30.480	242.000	14.7157	11.2380	86.93
185.32	30.480	244.000	14.4714	11.2695	87.19
185.32	30.480	246.000	14.2030	11.2762	87.45
185.32	30.480	248.000	13.9125	11.2573	87.72
185.32	30.480	250.000	13.6012	11.2123	87.98
185.32	30.480	252.000	13.2705	11.1408	88.24
185.32	30.480	254.000	12.9217	11.0426	88.50
185.32	30.480	256.000	12.5556	10.9175	88.75
185.32	30.480	258.000	12.1732	10.7657	89.01
185.32	30.480	260.000	11.7751	10.5872	89.27
185.32	30.480	262.000	11.3621	10.3823	89.52
185.32	30.480	264.000	10.9348	10.1512	89.78
185.32	30.480	266.000	10.4938	9.8944	90.03
185.32	30.480	268.000	10.0396	9.6123	90.28
185.32	30.480	270.000	9.5730	9.3056	90.54
185.32	30.480	272.000	9.0947	8.9748	90.79
185.32	30.480	274.000	8.6055	8.6207	91.04
185.32	30.480	276.000	8.1063	8.2441	91.28
185.32	30.480	278.000	7.5984	7.8460	91.53
185.32	30.480	280.000	7.0830	7.4275	91.78
185.32	30.480	282.000	6.5616	6.9898	92.02
185.32	30.480	284.000	6.0360	6.5343	92.27
185.32	30.480	286.000	5.5083	6.0627	92.51
185.32	30.480	288.000	4.9808	5.5769	92.76
185.32	30.480	290.000	4.4562	5.0791	93.00
185.32	30.480	292.000	3.9376	4.5719	93.24
185.32	30.480	294.000	3.4283	4.0583	93.48
185.32	30.480	296.000	2.9322	3.5418	93.72
185.32	30.480	298.000	2.4538	3.0268	93.96
185.32	30.480	300.000	1.9977	2.5180	94.20
185.32	30.480	302.000	1.5692	2.0214	94.44
185.32	30.480	304.000	1.1743	1.5434	94.68
185.32	30.480	306.000	0.8187	1.0913	94.91
185.32	30.480	308.000	0.5035	0.6678	95.15
185.32	30.480	310.000	0.2255	0.2716	95.38
185.32	30.480	312.000	-0.0186	-0.0987	95.62
185.32	30.480	314.000	-0.2324	-0.4445	95.85
185.32	30.480	316.000	-0.4191	-0.7674	96.08
185.32	30.480	318.000	-0.5820	-1.0689	96.31
185.32	30.480	320.000	-0.7240	-1.3504	96.54
185.32	30.480	322.000	-0.8478	-1.6134	96.77
185.32	30.480	324.000	-0.9559	-1.8591	97.00
185.32	30.480	326.000	-1.0504	-2.0890	97.23
185.32	30.480	328.000	-1.1333	-2.3042	97.46
185.32	30.480	330.000	-1.2063	-2.5058	97.69
185.32	30.480	332.000	-1.2708	-2.6950	97.91
185.32	30.480	334.000	-1.3282	-2.8728	98.14

185.32	30.480	336.000	-1.3794	-3.0399	98.37
185.32	30.480	338.000	-1.4254	-3.1974	98.59
185.32	30.480	340.000	-1.4671	-3.3459	98.81
185.32	30.480	342.000	-1.5051	-3.4862	99.04
185.32	30.480	344.000	-1.5401	-3.6190	99.26
185.32	30.480	346.000	-1.5724	-3.7447	99.48
185.32	30.480	348.000	-1.6026	-3.8640	99.70
185.32	30.480	350.000	-1.6309	-3.9773	99.92
185.32	30.480	352.000	-1.6578	-4.0851	100.14
185.32	30.480	354.000	-1.6835	-4.1878	100.36
185.32	30.480	356.000	-1.7081	-4.2858	100.58
185.32	30.480	358.000	-1.7319	-4.3793	100.80
185.32	30.480	360.000	-1.7551	-4.4686	101.02
185.32	30.480	362.000	-1.7778	-4.5545	101.24
185.32	30.480	364.000	-1.8000	-4.6366	101.45
185.32	30.480	366.000	-1.8220	-4.7155	101.67
185.32	30.480	368.000	-1.8438	-4.7912	101.88
185.32	30.480	370.000	-1.8654	-4.8640	102.10
185.32	30.480	372.000	-1.8869	-4.9341	102.31
185.32	30.480	374.000	-1.9084	-5.0017	102.53
185.32	30.480	376.000	-1.9299	-5.0669	102.74
185.32	30.480	378.000	-1.9514	-5.1298	102.95
185.32	30.480	380.000	-1.9731	-5.1906	103.16
185.32	30.480	382.000	-1.9948	-5.2494	103.37
185.32	30.480	384.000	-2.0166	-5.3063	103.58
185.32	30.480	386.000	-2.0387	-5.3614	103.80
185.32	30.480	388.000	-2.0608	-5.4148	104.00
185.32	30.480	390.000	-2.0832	-5.4666	104.21
185.32	30.480	392.000	-2.1057	-5.5169	104.42
185.32	30.480	394.000	-2.1284	-5.5658	104.63
185.32	30.480	396.000	-2.1513	-5.6133	104.84
185.32	30.480	398.000	-2.1744	-5.6594	105.04
185.32	30.480	400.000	-2.1977	-5.7043	105.25
185.32	30.480	402.000	-2.2212	-5.7481	105.46
185.32	30.480	404.000	-2.2449	-5.7907	105.66
185.32	30.480	406.000	-2.2686	-5.8320	105.87
185.32	30.480	408.000	-2.2928	-5.8725	106.07
185.32	30.480	410.000	-2.3171	-5.9120	106.28
185.32	30.480	412.000	-2.3417	-5.9505	106.48
185.32	30.480	414.000	-2.3664	-5.9881	106.68
185.32	30.480	416.000	-2.3914	-6.0249	106.88
185.32	30.480	418.000	-2.4166	-6.0608	107.09
185.32	30.480	420.000	-2.4419	-6.0959	107.29
185.32	30.480	422.000	-2.4675	-6.1302	107.49
185.32	30.480	424.000	-2.4933	-6.1638	107.69
185.32	30.480	426.000	-2.5192	-6.1967	107.89
185.32	30.480	428.000	-2.5454	-6.2288	108.09
185.32	30.480	430.000	-2.5717	-6.2603	108.29
185.32	30.480	432.000	-2.5983	-6.2912	108.49
185.32	30.480	434.000	-2.6250	-6.3214	108.69
185.32	30.480	436.000	-2.6520	-6.3510	108.88
185.32	30.480	438.000	-2.6791	-6.3801	109.08
185.32	30.480	440.000	-2.7065	-6.4086	109.28
185.32	30.480	442.000	-2.7340	-6.4365	109.47
185.32	30.480	444.000	-2.7617	-6.4640	109.67

185.32	30.480	446.000	-2.7896	-6.4909	109.87
185.32	30.480	448.000	-2.8176	-6.5173	110.06
185.32	30.480	450.000	-2.8459	-6.5433	110.25
185.32	30.480	452.000	-2.8743	-6.5688	110.45
185.32	30.480	454.000	-2.9030	-6.5938	110.64
185.32	30.480	456.000	-2.9318	-6.6184	110.84
185.32	30.480	458.000	-2.9607	-6.6426	111.03
185.32	30.480	460.000	-2.9899	-6.6664	111.22
185.32	30.480	462.000	-3.0192	-6.6898	111.41
185.32	30.480	464.000	-3.0488	-6.7128	111.61
185.32	30.480	466.000	-3.0784	-6.7354	111.80
185.32	30.480	468.000	-3.1083	-6.7576	111.99
185.32	30.480	470.000	-3.1384	-6.7795	112.18
185.32	30.480	472.000	-3.1686	-6.8011	112.37
185.32	30.480	474.000	-3.1990	-6.8223	112.56
185.32	30.480	476.000	-3.2295	-6.8432	112.75
185.32	30.480	478.000	-3.2603	-6.8638	112.94
185.32	30.480	480.000	-3.2912	-6.8840	113.12
185.32	30.480	482.000	-3.3222	-6.9040	113.31
185.32	30.480	484.000	-3.3535	-6.9236	113.50
185.32	30.480	486.000	-3.3849	-6.9430	113.69
185.32	30.480	488.000	-3.4165	-6.9621	113.87
185.32	30.480	490.000	-3.4482	-6.9809	114.06
185.32	30.480	492.000	-3.4801	-6.9994	114.25
185.32	30.480	494.000	-3.5122	-7.0177	114.43
185.32	30.480	496.000	-3.5445	-7.0357	114.62
185.32	30.480	498.000	-3.5769	-7.0535	114.80
185.32	30.480	500.000	-3.6094	-7.0710	114.99
185.32	30.480	502.000	-3.6422	-7.0883	115.17
185.32	30.480	504.000	-3.6750	-7.1053	115.36
185.32	30.480	506.000	-3.7081	-7.1222	115.54
185.32	30.480	508.000	-3.7413	-7.1388	115.72
185.32	30.480	510.000	-3.7747	-7.1551	115.90
185.32	30.480	512.000	-3.8082	-7.1713	116.09
185.32	30.480	514.000	-3.8419	-7.1873	116.27
185.32	30.480	516.000	-3.8757	-7.2030	116.45
185.32	30.480	518.000	-3.9097	-7.2186	116.63
185.32	30.480	520.000	-3.9438	-7.2339	116.81
185.32	30.480	522.000	-3.9781	-7.2491	116.99
185.32	30.480	524.000	-4.0125	-7.2640	117.17
185.32	30.480	526.000	-4.0471	-7.2788	117.35
185.32	30.480	528.000	-4.0818	-7.2934	117.53
185.32	30.480	530.000	-4.1167	-7.3078	117.71
185.32	30.480	532.000	-4.1517	-7.3221	117.89
185.32	30.480	534.000	-4.1869	-7.3362	118.07
185.32	30.480	536.000	-4.2222	-7.3501	118.25
185.32	30.480	538.000	-4.2576	-7.3638	118.43
185.32	30.480	540.000	-4.2932	-7.3774	118.60
185.32	30.480	542.000	-4.3289	-7.3908	118.78
185.32	30.480	544.000	-4.3648	-7.4041	118.96
185.32	30.480	546.000	-4.4008	-7.4172	119.14
185.32	30.480	548.000	-4.4369	-7.4301	119.31
185.32	30.480	550.000	-4.4731	-7.4429	119.49
185.32	30.480	552.000	-4.5095	-7.4556	119.66
185.32	30.480	554.000	-4.5460	-7.4681	119.84

185.32	30.480	556.000	-4.5826	-7.4805	120.01
185.32	30.480	558.000	-4.6194	-7.4927	120.19
185.32	30.480	560.000	-4.6563	-7.5049	120.36
185.32	30.480	562.000	-4.6933	-7.5168	120.54
185.32	30.480	564.000	-4.7304	-7.5287	120.71
185.32	30.480	566.000	-4.7676	-7.5404	120.88
185.32	30.480	568.000	-4.8050	-7.5520	121.06
185.32	30.480	570.000	-4.8424	-7.5634	121.23
185.32	30.480	572.000	-4.8800	-7.5748	121.40
185.32	30.480	574.000	-4.9177	-7.5860	121.58
185.32	30.480	576.000	-4.9555	-7.5971	121.75
185.32	30.480	578.000	-4.9934	-7.6081	121.92
185.32	30.480	580.000	-5.0314	-7.6189	122.09
185.32	30.480	582.000	-5.0695	-7.6297	122.26
185.32	30.480	584.000	-5.1077	-7.6403	122.43
185.32	30.480	586.000	-5.1461	-7.6508	122.60
185.32	30.480	588.000	-5.1845	-7.6613	122.77
185.32	30.480	590.000	-5.2230	-7.6716	122.94
185.32	30.480	592.000	-5.2616	-7.6818	123.11
185.32	30.480	594.000	-5.3003	-7.6919	123.28
185.32	30.480	596.000	-5.3391	-7.7019	123.45
185.32	30.480	598.000	-5.3780	-7.7118	123.62
185.32	30.480	600.000	-5.4169	-7.7216	123.79
185.32	30.480	602.000	-5.4560	-7.7313	123.96
185.32	30.480	604.000	-5.4951	-7.7409	124.13
185.32	30.480	606.000	-5.5343	-7.7504	124.29
185.32	30.480	608.000	-5.5736	-7.7599	124.46
185.32	30.480	610.000	-5.6130	-7.7692	124.63
185.32	30.480	612.000	-5.6525	-7.7784	124.79
185.32	30.480	614.000	-5.6920	-7.7876	124.96
185.32	30.480	616.000	-5.7316	-7.7966	125.13
185.32	30.480	618.000	-5.7713	-7.8056	125.29
185.32	30.480	620.000	-5.8110	-7.8145	125.46
185.32	30.480	622.000	-5.8509	-7.8233	125.62
185.32	30.480	624.000	-5.8907	-7.8320	125.79
185.32	30.480	626.000	-5.9307	-7.8407	125.95
185.32	30.480	628.000	-5.9707	-7.8492	126.12
185.32	30.480	630.000	-6.0107	-7.8577	126.28
185.32	30.480	632.000	-6.0509	-7.8661	126.45
185.32	30.480	634.000	-6.0911	-7.8744	126.61
185.32	30.480	636.000	-6.1313	-7.8826	126.78
185.32	30.480	638.000	-6.1716	-7.8908	126.94
185.32	30.480	640.000	-6.2119	-7.8989	127.10
185.32	30.480	642.000	-6.2523	-7.9069	127.26
185.32	30.480	644.000	-6.2928	-7.9148	127.43
185.32	30.480	646.000	-6.3333	-7.9227	127.59
185.32	30.480	648.000	-6.3738	-7.9305	127.75
185.32	30.480	650.000	-6.4144	-7.9382	127.91
185.32	30.480	652.000	-6.4550	-7.9459	128.08
185.32	30.480	654.000	-6.4957	-7.9535	128.24
185.32	30.480	656.000	-6.5364	-7.9610	128.40
185.32	30.480	658.000	-6.5772	-7.9684	128.56
185.32	30.480	660.000	-6.6180	-7.9758	128.72
185.32	30.480	662.000	-6.6588	-7.9831	128.88
185.32	30.480	664.000	-6.6996	-7.9904	129.04

185.32	30.480	666.000	-6.7405	-7.9976	129.20
185.32	30.480	668.000	-6.7814	-8.0047	129.36
185.32	30.480	670.000	-6.8224	-8.0118	129.52
185.32	30.480	672.000	-6.8634	-8.0188	129.68
185.32	30.480	674.000	-6.9044	-8.0257	129.84
185.32	30.480	676.000	-6.9454	-8.0326	130.00
185.32	30.480	678.000	-6.9864	-8.0394	130.15
185.32	30.480	680.000	-7.0275	-8.0461	130.31
185.32	30.480	682.000	-7.0686	-8.0528	130.47
185.32	30.480	684.000	-7.1097	-8.0595	130.63
185.32	30.480	686.000	-7.1508	-8.0661	130.79
185.32	30.480	688.000	-7.1920	-8.0726	130.94
185.32	30.480	690.000	-7.2332	-8.0791	131.10
185.32	30.480	692.000	-7.2743	-8.0855	131.26
185.32	30.480	694.000	-7.3155	-8.0918	131.41
185.32	30.480	696.000	-7.3567	-8.0981	131.57
185.32	30.480	698.000	-7.3979	-8.1044	131.73
185.32	30.480	700.000	-7.4392	-8.1106	131.88
185.32	30.480	702.000	-7.4804	-8.1167	132.04
185.32	30.480	704.000	-7.5216	-8.1228	132.19
185.32	30.480	706.000	-7.5629	-8.1289	132.35
185.32	30.480	708.000	-7.6041	-8.1349	132.50
185.32	30.480	710.000	-7.6454	-8.1408	132.66
185.32	30.480	712.000	-7.6867	-8.1467	132.81
185.32	30.480	714.000	-7.7279	-8.1525	132.97
185.32	30.480	716.000	-7.7692	-8.1583	133.12
185.32	30.480	718.000	-7.8104	-8.1641	133.28
185.32	30.480	720.000	-7.8517	-8.1698	133.43
185.32	30.480	722.000	-7.8930	-8.1754	133.58
185.32	30.480	724.000	-7.9342	-8.1810	133.74
185.32	30.480	726.000	-7.9755	-8.1866	133.89
185.32	30.480	728.000	-8.0167	-8.1921	134.04
185.32	30.480	730.000	-8.0580	-8.1975	134.20
185.32	30.480	732.000	-8.0992	-8.2030	134.35
185.32	30.480	734.000	-8.1404	-8.2083	134.50
185.32	30.480	736.000	-8.1816	-8.2137	134.65
185.32	30.480	738.000	-8.2228	-8.2190	134.81
185.32	30.480	740.000	-8.2640	-8.2242	134.96
185.32	30.480	742.000	-8.3052	-8.2294	135.11
185.32	30.480	744.000	-8.3464	-8.2345	135.26
185.32	30.480	746.000	-8.3876	-8.2397	135.41
185.32	30.480	748.000	-8.4287	-8.2447	135.56
185.32	30.480	750.000	-8.4698	-8.2498	135.71
185.32	30.480	752.000	-8.5109	-8.2548	135.86
185.32	30.480	754.000	-8.5520	-8.2597	136.01
185.32	30.480	756.000	-8.5931	-8.2646	136.16
185.32	30.480	758.000	-8.6342	-8.2695	136.31
185.32	30.480	760.000	-8.6752	-8.2743	136.46
185.32	30.480	762.000	-8.7163	-8.2791	136.61
185.32	30.480	764.000	-8.7573	-8.2838	136.76
185.32	30.480	766.000	-8.7982	-8.2886	136.91
185.32	30.480	768.000	-8.8392	-8.2932	137.06
185.32	30.480	770.000	-8.8801	-8.2979	137.21
185.32	30.480	772.000	-8.9211	-8.3025	137.36
185.32	30.480	774.000	-8.9619	-8.3070	137.51

185.32	30.480	776.000	-9.0028	-8.3116	137.65
185.32	30.480	778.000	-9.0437	-8.3160	137.80
185.32	30.480	780.000	-9.0845	-8.3205	137.95
185.32	30.480	782.000	-9.1253	-8.3249	138.10
185.32	30.480	784.000	-9.1660	-8.3293	138.24
185.32	30.480	786.000	-9.2068	-8.3336	138.39
185.32	30.480	788.000	-9.2475	-8.3379	138.54
185.32	30.480	790.000	-9.2882	-8.3422	138.69
185.32	30.480	792.000	-9.3288	-8.3465	138.83
185.32	30.480	794.000	-9.3695	-8.3507	138.98
185.32	30.480	796.000	-9.4101	-8.3548	139.12
185.32	30.480	798.000	-9.4506	-8.3590	139.27
185.32	30.480	800.000	-9.4912	-8.3631	139.42
185.32	30.480	802.000	-9.5317	-8.3672	139.56
185.32	30.480	804.000	-9.5722	-8.3712	139.71
185.32	30.480	806.000	-9.6126	-8.3752	139.85
185.32	30.480	808.000	-9.6530	-8.3792	140.00
185.32	30.480	810.000	-9.6934	-8.3831	140.14
185.32	30.480	812.000	-9.7338	-8.3871	140.29
185.32	30.480	814.000	-9.7741	-8.3909	140.43
185.32	30.480	816.000	-9.8144	-8.3948	140.58
185.32	30.480	818.000	-9.8547	-8.3986	140.72
185.32	30.480	820.000	-9.8949	-8.4024	140.87
185.32	30.480	822.000	-9.9351	-8.4061	141.01
185.32	30.480	824.000	-9.9753	-8.4099	141.15
185.32	30.480	826.000	-10.0154	-8.4136	141.30
185.32	30.480	828.000	-10.0555	-8.4172	141.44
185.32	30.480	830.000	-10.0956	-8.4209	141.58
185.32	30.480	832.000	-10.1356	-8.4245	141.73
185.32	30.480	834.000	-10.1756	-8.4281	141.87
185.32	30.480	836.000	-10.2156	-8.4316	142.01
185.32	30.480	838.000	-10.2556	-8.4351	142.15
185.32	30.480	840.000	-10.2955	-8.4386	142.30
185.32	30.480	842.000	-10.3354	-8.4421	142.44
185.32	30.480	844.000	-10.3752	-8.4455	142.58
185.32	30.480	846.000	-10.4151	-8.4490	142.72
185.32	30.480	848.000	-10.4549	-8.4523	142.87
185.32	30.480	850.000	-10.4946	-8.4557	143.01
185.32	30.480	852.000	-10.5344	-8.4590	143.15
185.32	30.480	854.000	-10.5741	-8.4623	143.29
185.32	30.480	856.000	-10.6138	-8.4656	143.43
185.32	30.480	858.000	-10.6534	-8.4689	143.57
185.32	30.480	860.000	-10.6931	-8.4721	143.71
185.32	30.480	862.000	-10.7327	-8.4753	143.85
185.32	30.480	864.000	-10.7723	-8.4784	143.99
185.32	30.480	866.000	-10.8118	-8.4816	144.13
185.32	30.480	868.000	-10.8514	-8.4847	144.27
185.32	30.480	870.000	-10.8909	-8.4878	144.41
185.32	30.480	872.000	-10.9304	-8.4909	144.55
185.32	30.480	874.000	-10.9699	-8.4939	144.69
185.32	30.480	876.000	-11.0094	-8.4969	144.83
185.32	30.480	878.000	-11.0488	-8.4999	144.97
185.32	30.480	880.000	-11.0882	-8.5029	145.11
185.32	30.480	882.000	-11.1276	-8.5059	145.25
185.32	30.480	884.000	-11.1670	-8.5088	145.39

185.32	30.480	886.000	-11.2064	-8.5117	145.53
185.32	30.480	888.000	-11.2458	-8.5145	145.66
185.32	30.480	890.000	-11.2852	-8.5174	145.80
185.32	30.480	892.000	-11.3245	-8.5202	145.94
185.32	30.480	894.000	-11.3639	-8.5230	146.08
185.32	30.480	896.000	-11.4032	-8.5258	146.22
185.32	30.480	898.000	-11.4426	-8.5286	146.36
185.32	30.480	900.000	-11.4819	-8.5313	146.49
185.32	30.480	902.000	-11.5212	-8.5340	146.63
185.32	30.480	904.000	-11.5606	-8.5367	146.77
185.32	30.480	906.000	-11.5999	-8.5394	146.90
185.32	30.480	908.000	-11.6393	-8.5420	147.04
185.32	30.480	910.000	-11.6786	-8.5446	147.18
185.32	30.480	912.000	-11.7180	-8.5472	147.31
185.32	30.480	914.000	-11.7573	-8.5498	147.45
185.32	30.480	916.000	-11.7967	-8.5524	147.59
185.32	30.480	918.000	-11.8361	-8.5549	147.72
185.32	30.480	920.000	-11.8755	-8.5574	147.86
185.32	30.480	922.000	-11.9150	-8.5599	148.00
185.32	30.480	924.000	-11.9544	-8.5624	148.13
185.32	30.480	926.000	-11.9939	-8.5648	148.27
185.32	30.480	928.000	-12.0334	-8.5673	148.40
185.32	30.480	930.000	-12.0729	-8.5697	148.54
185.32	30.480	932.000	-12.1125	-8.5721	148.67
185.32	30.480	934.000	-12.1521	-8.5744	148.81
185.32	30.480	936.000	-12.1917	-8.5768	148.94
185.32	30.480	938.000	-12.2313	-8.5791	149.08
185.32	30.480	940.000	-12.2710	-8.5814	149.21
185.32	30.480	942.000	-12.3108	-8.5837	149.35
185.32	30.480	944.000	-12.3506	-8.5860	149.48
185.32	30.480	946.000	-12.3904	-8.5882	149.62
185.32	30.480	948.000	-12.4303	-8.5904	149.75
185.32	30.480	950.000	-12.4703	-8.5926	149.88
185.32	30.480	952.000	-12.5103	-8.5948	150.02
185.32	30.480	954.000	-12.5503	-8.5970	150.15
185.32	30.480	956.000	-12.5905	-8.5992	150.28
185.32	30.480	958.000	-12.6307	-8.6013	150.42
185.32	30.480	960.000	-12.6709	-8.6034	150.55
185.32	30.480	962.000	-12.7113	-8.6055	150.68
185.32	30.480	964.000	-12.7517	-8.6076	150.82
185.32	30.480	966.000	-12.7922	-8.6096	150.95
185.32	30.480	968.000	-12.8328	-8.6117	151.08
185.32	30.480	970.000	-12.8735	-8.6137	151.21
185.32	30.480	972.000	-12.9142	-8.6157	151.35
185.32	30.480	974.000	-12.9551	-8.6177	151.48
185.32	30.480	976.000	-12.9960	-8.6196	151.61
185.32	30.480	978.000	-13.0371	-8.6216	151.74
185.32	30.480	980.000	-13.0782	-8.6235	151.87
185.32	30.480	982.000	-13.1195	-8.6254	152.01
185.32	30.480	984.000	-13.1609	-8.6273	152.14
185.32	30.480	986.000	-13.2024	-8.6292	152.27
185.32	30.480	988.000	-13.2440	-8.6311	152.40
185.32	30.480	990.000	-13.2858	-8.6329	152.53
185.32	30.480	992.000	-13.3277	-8.6347	152.66
185.32	30.480	994.000	-13.3697	-8.6365	152.79

185.32	30.480	996.000	-13.411A	-8.63A3	152.92
185.32	30.480	998.000	-13.4541	-8.6401	153.05
185.32	30.480	1000.000	-13.4965	-8.6419	153.19
185.32	30.480	1002.000	-13.5391	-8.6436	153.32
185.32	30.480	1004.000	-13.5819	-8.6453	153.45
185.32	30.480	1006.000	-13.624A	-8.6470	153.58
185.32	30.480	1008.000	-13.667A	-8.6487	153.71
185.32	30.480	1010.000	-13.7111	-8.6504	153.84
185.32	30.480	1012.000	-13.7545	-8.6521	153.97
185.32	30.480	1014.000	-13.79A1	-8.6537	154.09
185.32	30.480	1016.000	-13.8418	-8.6553	154.22
185.32	30.480	1018.000	-13.885A	-8.6570	154.35
185.32	30.480	1020.000	-13.9300	-8.6585	154.48
185.32	30.480	1022.000	-13.9743	-8.6601	154.61
185.32	30.480	1024.000	-14.0189	-8.6617	154.74
185.32	30.480	1026.000	-14.0636	-8.6632	154.87
185.32	30.480	1028.000	-14.10A6	-8.6648	155.00
185.32	30.480	1030.000	-14.153A	-8.6663	155.13
185.32	30.480	1032.000	-14.1992	-8.6678	155.26
185.32	30.480	1034.000	-14.244A	-8.6693	155.38
185.32	30.480	1036.000	-14.2907	-8.6707	155.51
185.32	30.480	1038.000	-14.336A	-8.6722	155.64
185.32	30.480	1040.000	-14.3832	-8.6736	155.77
185.32	30.480	1042.000	-14.4298	-8.6751	155.90
185.32	30.480	1044.000	-14.4767	-8.6765	156.02
185.32	30.480	1046.000	-14.523A	-8.6779	156.15
185.32	30.480	1048.000	-14.5712	-8.6793	156.28
185.32	30.480	1050.000	-14.6189	-8.6A06	156.41
185.32	30.480	1052.000	-14.666A	-8.6A20	156.53
185.32	30.480	1054.000	-14.7150	-8.6A33	156.66
185.32	30.480	1056.000	-14.7636	-8.6A46	156.79
185.32	30.480	1058.000	-14.8124	-8.6A59	156.91
185.32	30.480	1060.000	-14.8615	-8.6A72	157.04
185.32	30.480	1062.000	-14.9109	-8.6A85	157.17
185.32	30.480	1064.000	-14.9606	-8.6A98	157.29
185.32	30.480	1066.000	-15.0107	-8.6910	157.42
185.32	30.480	1068.000	-15.0611	-8.6923	157.55
185.32	30.480	1070.000	-15.111A	-8.6935	157.67
185.32	30.480	1072.000	-15.1629	-8.6947	157.80
185.32	30.480	1074.000	-15.2143	-8.6959	157.92
185.32	30.480	1076.000	-15.2660	-8.6971	158.05
185.32	30.480	1078.000	-15.31A1	-8.6983	158.18
185.32	30.480	1080.000	-15.3706	-8.6994	158.30
185.32	30.480	1082.000	-15.4235	-8.7006	158.43
185.32	30.480	1084.000	-15.4767	-8.7017	158.55
185.32	30.480	1086.000	-15.5303	-8.7028	158.68
185.32	30.480	1088.000	-15.5843	-8.7039	158.80
185.32	30.480	1090.000	-15.6387	-8.7050	158.93
185.32	30.480	1092.000	-15.6936	-8.7061	159.05
185.32	30.480	1094.000	-15.74A8	-8.7071	159.18
185.32	30.480	1096.000	-15.8044	-8.7082	159.30
185.32	30.480	1098.000	-15.8605	-8.7092	159.43
185.32	30.480	1100.000	-15.9170	-8.7102	159.55
185.32	30.480	1102.000	-15.9740	-8.7112	159.67
185.32	30.480	1104.000	-16.0314	-8.7122	159.80

185.32	30.480	1106.000	-16.0893	-8.7132	159.92
185.32	30.480	1108.000	-16.1476	-8.7142	160.05
185.32	30.480	1110.000	-16.2064	-8.7151	160.17
185.32	30.480	1112.000	-16.2658	-8.7161	160.29
185.32	30.480	1114.000	-16.3255	-8.7170	160.42
185.32	30.480	1116.000	-16.3858	-8.7179	160.54
185.32	30.480	1118.000	-16.4466	-8.7189	160.67
185.32	30.480	1120.000	-16.5080	-8.7197	160.79
185.32	30.480	1122.000	-16.5698	-8.7206	160.91
185.32	30.480	1124.000	-16.6322	-8.7215	161.03
185.32	30.480	1126.000	-16.6951	-8.7224	161.16
185.32	30.480	1128.000	-16.7586	-8.7232	161.28
185.32	30.480	1130.000	-16.8227	-8.7240	161.40
185.32	30.480	1132.000	-16.8873	-8.7249	161.53
185.32	30.480	1134.000	-16.9525	-8.7257	161.65
185.32	30.480	1136.000	-17.0183	-8.7265	161.77
185.32	30.480	1138.000	-17.0847	-8.7273	161.89
185.32	30.480	1140.000	-17.1517	-8.7280	162.02
185.32	30.480	1142.000	-17.2193	-8.7288	162.14
185.32	30.480	1144.000	-17.2876	-8.7296	162.26
185.32	30.480	1146.000	-17.3565	-8.7303	162.38
185.32	30.480	1148.000	-17.4261	-8.7310	162.50
185.32	30.480	1150.000	-17.4963	-8.7317	162.62
185.32	30.480	1152.000	-17.5672	-8.7324	162.75
185.32	30.480	1154.000	-17.6388	-8.7331	162.87
185.32	30.480	1156.000	-17.7112	-8.7338	162.99
185.32	30.480	1158.000	-17.7842	-8.7345	163.11
185.32	30.480	1160.000	-17.8579	-8.7352	163.23
185.32	30.480	1162.000	-17.9325	-8.7358	163.35
185.32	30.480	1164.000	-18.0077	-8.7364	163.47
185.32	30.480	1166.000	-18.0837	-8.7371	163.59
185.32	30.480	1168.000	-18.1605	-8.7377	163.72
185.32	30.480	1170.000	-18.2382	-8.7383	163.84
185.32	30.480	1172.000	-18.3166	-8.7389	163.96
185.32	30.480	1174.000	-18.3958	-8.7395	164.08
185.32	30.480	1176.000	-18.4759	-8.7400	164.20
185.32	30.480	1178.000	-18.5569	-8.7406	164.32
185.32	30.480	1180.000	-18.6387	-8.7411	164.44
185.32	30.480	1182.000	-18.7214	-8.7417	164.56
185.32	30.480	1184.000	-18.8050	-8.7422	164.68
185.32	30.480	1186.000	-18.8896	-8.7427	164.80
185.32	30.480	1188.000	-18.9751	-8.7432	164.92
185.32	30.480	1190.000	-19.0615	-8.7437	165.04
185.32	30.480	1192.000	-19.1490	-8.7442	165.16
185.32	30.480	1194.000	-19.2374	-8.7447	165.28
185.32	30.480	1196.000	-19.3269	-8.7451	165.39
185.32	30.480	1198.000	-19.4174	-8.7456	165.51
185.32	30.480	1200.000	-19.5090	-8.7460	165.63
185.32	30.480	1202.000	-19.6017	-8.7465	165.75
185.32	30.480	1204.000	-19.6955	-8.7469	165.87
185.32	30.480	1206.000	-19.7904	-8.7473	165.99
185.32	30.480	1208.000	-19.8865	-8.7477	166.11
185.32	30.480	1210.000	-19.9838	-8.7481	166.23
185.32	30.480	1212.000	-20.0823	-8.7485	166.35
185.32	30.480	1214.000	-20.1821	-8.7489	166.46
185.32	30.480	1216.000	-20.2831	-8.7492	166.58
185.32	30.480	1218.000	-20.3854	-8.7496	166.70
185.32	30.480	1220.000	-20.4890	-8.7499	166.82

11. REMARKS

XWVG has been run for cases involving over 50 different trilinear modified refractivity profiles and for frequencies between 50 MHz and 94 GHz. In some of these cases modes with attenuation rates up to 3 dB/km were found. In every case XWVG ran without problems. XWVG works reliably in the diffraction region and well into the line-of-sight region. The results of XWVG were compared to results from other waveguide programs in the diffraction region and to results from a ray trace program in the line-of-sight region. The comparison was very good in the diffraction region and well into the line-of-sight region.

For elevated layers XWVG's strength is in the frequency range of 50 to several hundred MHz. Here the diffraction type modes have attenuation rates as low as a few tenths of a dB per kilometer and could therefore affect the signal level value. XWVG easily finds these diffraction type modes while "G trace" type waveguide programs [6] may miss them. Also, for these frequencies the number of modes with attenuation rates below, for example, 1 dB/km is small. The region of the complex q_{11}-plane that has to be searched for these modes is also small. Therefore, the computer time required by XWVG is not excessively long. At higher frequencies the number of modes below a given attenuation rate increases. This increase in the number of modes comes with an increase in the region of the complex q_{11}-plane that has to be searched for modes. Therefore, the computer run time of XWVG increases, sometimes dramatically. At these higher frequencies the attenuation rate of the diffraction type modes increases, so the fact that G-trace type waveguide programs may fail to find them is no longer significant. In a sense XWVG is complementary to the G-trace type program of Pappert and Goodhart [6].

A major improvement to XWVG that is contemplated is the capability to handle modified refractivity profiles that have N linear segments, where N is a moderate sized number, e.g., 20. In order to do this, use will be made of an improved method for evaluating the derivative of a determinant [37, 38]. This extension of the capabilities of XWVG will be especially useful for multisegment evaporation duct models. It will also allow modeling of evaporation ducts in combination with elevated ducts. Since XWVG uses the

Shellman-Morfitt root-finding routine, all the modes with an attenuation rate less than some specified value are guaranteed to be found. This guarantee is something that multilayer extensions of the G-trace type program cannot give.

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